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A Historical and Critical Review
of Mathematical Science

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XXIV

INTEGRALS OF LEBESGUE. 1917

BY PROFESSOR GILBERT AMES BLISS.

(Read at the Chicago Symposium of the American Mathematical Society, April 6, 1917.)

purpose of this paper is to present in detail the contents ntroductory lecture given by the writer at the sym-um of the American Mathematical Society held in Chicago n April of this year on the subject of Lebesgue integrals. It would be impossible to have selected a subject for that occasion more characteristic of present mathematical tendencies. Volterra has pointed out, in the introductory chapter of his Leçons sur les Fonctions des Lignes, the rapid development which is taking place in our notions of infinite processes, examples of which are the definite integral limit, the solution of integral equations, and the transition from functions of a finite number of variables to functions of lines. In the field of integration the classical integral of Riemann, perfected by oux, was such a convenient and perfect instrument that d itself for a long time upon the mathematical eing something unique and final. The advent of the integrals of Stieltjes and Lebesgue has shaken the complacency of mathematicians in this respect, and, with the theory of linear integral equations, has given the signal for a reexamination and extension of many of the types of processes which Volterra calls passing from the finite to the infinite.

It should be noted that the Lebesgue integral is only one of the evidences of this restlessness in the particular domain of the integration theory. Other new definitions of an integral have been devised by Stieltjes, W. H. Young, Pierpont, Hellinger, Radon, Fréchet, E. H. Moore, and others. The definitions of Lebesgue, Young, and Pierpont, and those of Stieltjes and Hellinger, form two rather well defined and distinct types, while that of Radon is a generalization of the integrals of both Lebesgue and Stieltjes. The efforts of Fréchet and Moore have been directed toward definitions valid on more general ranges than sets of points of a line or higher spaces, and which include the others for special cases of these ranges. Lebesgue and Hahn, with the help of somewhat complicated transformations, have shown that the integrals of Stieltjes and Hellinger are expressible as Lebesgue integrals. It seems unfortunate that these reductions should have been interpreted by some as an attempt to establish the Lebesgue integral in the unique position so long occupied by the integral of Riemann. Van Vleck has in fact remarked* that a Lebesgue integral is expressible as one of Stieltjes by transformation much simpler than that used by L the opposite purpose, and the Stieltjes integral so readily expressible in terms of a Riemann integral, as in § 6 below. Furthermore the Stieltjes integral seems distinctly better suited than that of Lebesgue to certain types of questions, as is well indicated by the original "problem of moments" of Stieltjes,† or by a generalization of it which Riesz has made in his remarkable discussion of a problem analogous to the determination of a function whose Fourier constants are given.‡

The conclusion then seems to be that one should reserve judgment, for the present at least, as to the final form or forms which the integration theory is to take. It is probab that the outcome may be a general theory of the type of Fréchet and Moore, having not one but a nu instances with forms more adaptable to probl special types. However this may be, there can as to the wide influence which the work of Borel, Lebesgue, and their followers is having upon the mathematical thought of the present time, and no question as to the notable advances which have been made in the many domains of real function theory to which the Lebesgue form of integral is especially adapted. Not all of these can be recounted here, but it is hoped that the selection made will indicate the tendencies of the theory, and make the approach to its further results an e[illegible]e.

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* "Haskins's momental theorem and its connection with Stieltjes's problem of moments," *Transactions of the American Mathematical Society*, vol. 18 (1917), p. 327.

† "Recherches sur les fractions continues," *Annales de la Faculté des Sciences de Toulouse*, vol. 8 (1894).

‡ Riesz. "Sur certaines systèmes singuliers d'équations intégrales," *Annaels Scientifiques de l'Ecole Normale Supérieure*, vol. 28 (1911), p. 33.

[illegible]eory of Lebesgue integrals has had an able expounder in the person of de la Vallée Poussin, who has published systematic treatments of portions of it at some five different times and places, indicated in the list of references at the end of this paper. It is characteristic of the formative state [illegible]he subject at the present time that each of his presenta[illegible] has signalized new theorems, new points of view, or [illegible]ved methods of proof. The account given in his recent [illegible]es at the Collège de France, designated by the number [illegible]the reference list at the end of this paper, is especially [illegible] and satisfying, though somewhat more sophisticated [illegible] of his earlier disquisitions.

[illegible]est device of de la Vallée Poussin in approaching the [illegible] of measure is to develop first of all the theory of the measures of denumerable sums of intervals and of closed sets of points, and then to define in terms of them the measures of more general sets.* A closed and bounded set is always the portion of an interval remaining after the extraction of the interiors of a denumerable sum of non-overlapping intervals. I have been interested here to attempt to return again to the more direct methods of Borel and Lebesgue,† which found the theory of measure entirely upon the measurability of denumerable sets of intervals. With the aid of the improved methods [illegible] proof devised by de la Vallée Poussin it is possible to [illegible]h the subject in a way which seems to me to be espe[illegible] and clear, and possible at the same time to [illegible]ut added complication the foundations of the [illegible] positive additive functions of a point set, of which the measure function is a special case.

The notion of a function $g(e)$ real, single-valued, and additive on a class of point sets e, was first emphasized by Lebesgue‡ in his search for something which might be called the indefinite integral of a function of several variables, and the theory of such functions of a point set was subsequently much perfected by de la Vallée Poussin[*] and Radon.[†] A function $g(e)$ is said to be continuous if its value approaches zero with the diameter of e, and it is absolutely continuous if its value approaches zero with the measure of e. Every fun[illegible] wh[illegible] is absolutely continuous is clearly continuous, a[illegible] a waste of effort therefore to develop their common [illegible] separately, as has usually been done. It turns o[illegible]t is as easy to proceed directly to the theory of functi[illegible]ch are only continuous, as it is to found the theory of c[illegible]uous functions upon the absolutely continuous case. A number of the properties of these functions which de la Vallée Poussin establishes independently in order to prove th[illegible]mul[illegible] of § 8 are immediate consequences of this formula w[illegible] proved directly.

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* VII, Chapitre II.

† I, p. 46; II, p. 238; III, p. 106.

‡ IV, p. 361.

1. *Definition and Existence of Lebesgue Integr*[illegible]

The relation between the integral of Lebesgue a[illegible]as-sical integral of Riemann may be seen somewhat [illegible]uitively by considering first a function $f(x)$ w[illegible]o[illegible]uous and has only a finite number of maxima a[illegible]a on an interval ab. Let l_k $(k = 0, 1, \cdots, n)$ be a s[illegible] values with the properties

$$(1) \qquad l_0 < \mu, \quad M < l_n, \quad 0 < l_k - l_{k-1} \leqq \epsilon \quad (k = 1, 2, \cdots, n),$$

where μ and M are the lower and upper bounds, respectively, of the values of $f(x)$ on ab. If the points l_{k-1}, l_k are marked on the y-axis, as in the figure, a corresponding set of points e_k is determined on the x-axis consisting of all the abscissas x for which the values of $f(x)$ lie between l_{k-1} and l_k. Let η_k be a value arbitrarily selected on the interval $l_{k-1}l_k$, and let $m(e_k)$ denote the sum of the lengths of the x-intervals in the point set e_k. Then

$$(2) \qquad S = \sum_{k=1}^{n} \eta_k m(e_k) = \sum_{i=1}^{p} f(\xi_i)\Delta x_i,$$

where the totality of intervals Δx_i $(i = 1, 2, \cdots, p)$ is that defined by the sets e_k $(k = 1, 2, \cdots, n)$, and where ξ_i is chosen in each interval Δx_i of e_k so that $f(\xi_i) = \eta_k$. The maximum δ of the lengths of the intervals Δx_i will approach zero with ϵ, and it follows that

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[*] VI, and VII, Chapitre IV.

[†] VIII, p. 1299.

$$\lim_{\epsilon=0}\sum_{k=1}^{n}\eta_k m(e_k) = \lim_{\delta=0}\sum_{i=1}^{p} f(\xi_i)\Delta x_i = \int^{b} f(x)dx.$$

The [illegible]ly value of this argument for the purposes of the [illegible]er is to show how the usual definite integral limit [illegible]sy o[illegible]btained in a special case by a partition on the y-axis, aThe th[illegible]s by the usual subdivision of the x-interval ab. A d[illegible] at once presents itself when it is attempted to extend this [illegible]hod of procedure to functions not necessarily continuous or having a finite number of maxima and minima. For such less special functions the set of points e_k will not in [illegible]al co[illegible]t of intervals, and hence will not have a well-defined measure, unless some adequate definition of the measure of a point set has been previously set down.

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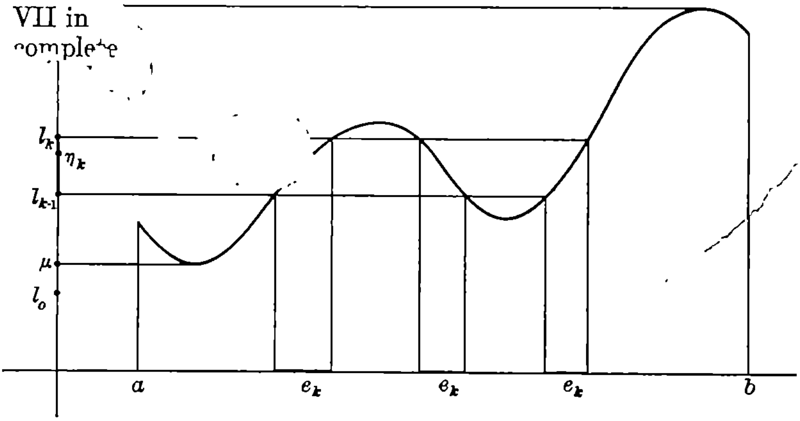

FIG. 1.

The notions of the measure of a point set and measurable functions, which lie at the root of the Lebesgue theory of integration, will be examined in detail in later sections. For the present suppose that a definition of measure has been given which may not apply to all point sets, but which does apply to some at least, and let such sets be called measurable sets. Further suppose that the measure $m(e)$ of a measurable set e is always positive or zero, and that when $e_1, e_2, \cdots, e_n$ are measurable and distinct, the set $e_1 + e_2 + \cdots + e_n$ consisting of all their points taken together is also measurable, and

$$m(e_1 + e_2 + \cdots + e_n) = m(e_1) + m(e_2) + \cdots + m(e_n).$$

Let $f(x)$ be well defined and bounded at the points of a set e, and denote by $e[A \leqq f(x) < B]$ the points of e where $A \leqq f(x) < B$. Then f is said to be measurable on e if the set $e[A \leqq f(x) < B]$ is measurable for every pair of constants A, B.

DEFINITION OF A LEBESGUE INTEGRAL.* *Let* $f(x)$ *be a real single valued function which is measurable and has bounds* μ, M *on a point set* e. *Select a set of values* l_k $(k = 0, 1, \cdots, n)$ *with the properties* (1), *and let* η_k *be a value anywhere on the interval* $l_{k-1}l_k$. *Denote the set of points* $e[l_{k-1} \leqq f(x) < l_k]$ by e_k. *Then by definition*

$$\int_e f(x)dx = \lim_{\epsilon=0} \sum_{k=1}^{n} \eta_k m(e_k). \tag{3}$$

To prove that the integral limit always exists under the circumstances described in the definition, consider first two sums S, S' like the one in the second member of (3), and suppose that the ladder of l-values for S' includes all of those of S. For simplicity of proof let the interval $l_{k-1}l_k$ of S be divided by only two points l', l'' in forming S', so that e_k is decomposed into three parts $e_k = e' + e'' + e'''$. Then the portions of S and S' corresponding to the interval $l_{k-1}l_k$ are

$$\begin{aligned} S:&\quad \eta_k m(e_k) = \eta_k m(e') + \eta_k m(e'') + \eta_k m(e'''),\\ S':&\quad \eta' m(e') + \eta'' m(e'') + \eta''' m(e'''), \end{aligned}$$

and they differ by less than ϵe_k since

$$|\eta_k - \eta'| < \epsilon, \quad |\eta_k - \eta''| < \epsilon, \quad |\eta_k - \eta'''| < \epsilon.$$

This result would be the same if there were more or less than two new values l', l''. It follows then that

$$|S - S'| < \sum_k \epsilon m(e_k) = \epsilon m(e).$$

The constant ϵ of the definition may be called a norm for the sum S. All sums of this type lie between $(\mu - \epsilon)m(e)$ and $(M + \epsilon)m(e)$. Hence out of an arbitrarily selected sequence of sums with norms approaching zero, a sub-sequence $\{S_n\}$

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* VII, p. 39; V, vol. 1, 3d edition, p. 257; III, p. 112.

may be chosen which has a limit I, and the corresponding sequence of norms $\{\epsilon_n\}$ will still have limit zero.

For every sum S with norm ϵ it follows then that

$$\begin{aligned} | S - I | &\leqq | S - S' | + | S' - S_n | + | S_n - I | \\ &< \epsilon m(e) + \epsilon_n m(e) + | S_n - I |, \end{aligned}$$

where S' is formed by using all the l-values of both S and S_n. But by taking ϵ sufficiently small and n sufficiently large, each term on the right can be made less than $\delta/3$, from which follows

THEOREM 1. *The Lebesgue integral limit* (3) *always exists for a function* $f(x)$ *which is measurable and bounded on the set* e.

§ 2. *Preliminary Properties of Point Sets.**

All of the point sets e to be considered are supposed to lie on a finite interval ab. It has been found convenient to define the sum $e_1 + e_2$ of two sets e_1, e_2 to be the totality of their points, the difference $e_1 - e_2$ to be the set of points which are in e_1 but not in e_2, and the product e_1e_2 to be the totality of points which e_1 and e_2 have in common. Addition and multiplication are readily seen to be commutative and associative, and to satisfy the relations

$$(e_1 + e_2)e_3 = e_1e_3 + e_2e_3, \quad (e_1 - e_2)e_3 = e_1e_3 - e_2e_3.$$

But one must be careful about subtraction. The sets $(e_1 + e_2) - e_3$ and $e_1 + (e_2 - e_3)$ are not necessarily the same, as a glance at the accompanying figure for plane sets will show.

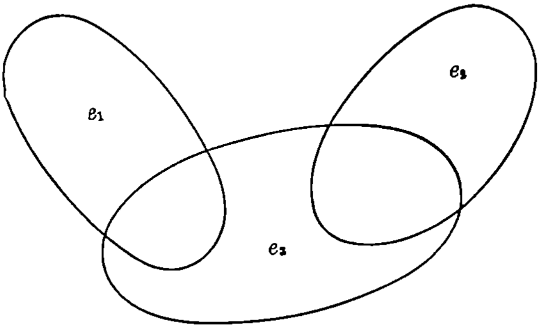

FIG. 2.

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* VII, pp. 5 18; V, vol. 1, 3d edition, p. 59.

The complement Ce of a set e is the totality of points of the interval ab which are not in e. The difference and product of two sets e_1, e_2 are expressible in terms of addition and complements. For

$$C(e_1 - e_2) = Ce_1 + e_2, \quad Ce_1e_2 = Ce_1 + Ce_2. \tag{4}$$

Hence theoretically the only operations which need to be considered are addition and the taking of complements. The others are, however, frequently found to be of great convenience.

The complete limit c of a sequence of sets $\{e_n\}$ is defined to be the totality of points each of which occurs in an infinity of the point sets e_n. It is representable by the formula

$$c = (e_1 + e_2 + \cdots)(e_2 + e_3 + \cdots)\cdots. \tag{5}$$

The restricted limit r is the set of points for each of which there is a place in the sequence $\{e_n\}$ beyond which the point occurs in every e_n. A formula for r is

$$r = (e_1e_2\cdots) + (e_2e_3\cdots) + \cdots. \tag{6}$$

The set r is always a part of the set c, and when the two are identical the sequence e_n is said to have a unique limit

$$\lim e_n = c = r. \tag{7}$$

A particular case of the limits c and r is illustrated graphically for plane sets in Fig. 3. If all the sets e_n of odd index are identical with the vertical rectangle in Fig. 3, and all of even index identical with the horizontal rectangle, the complete limit is the totality of points of both, and the restricted limit is the square common to the two rectangles. On the other hand, if the sets of odd index shrink successively as indicated by the dotted lines, and similarly for those of even index, the complete and restricted limits are both identical with the corner square, and the sequence $\{e_n\}$ has this square as its unique limit.

There are two special cases for which the limit of a sequence $\{e_n\}$ is easily seen to exist. These are when each element e_n is contained in the following, and when each e_n is contained in the preceding. The limits are then

$$\lim e_n = e_1 + e_2 + \cdots, \quad \lim e_n = e_1e_2\cdots,$$

respectively. It follows that for every sequence $\{e_n\}$ whatsoever

$$e_1 + e_2 + \cdots = \lim (e_1 + \cdots + e_n), \quad e_1 e_2 \cdots = \lim (e_1 e_2 \cdots e_n),$$

and furthermore from formulas (5) and (6)

$$(8) \qquad c = \lim (e_n + e_{n+1} \cdots), \quad r = \lim (e_n e_{n+1} \cdots).$$

Lemma 1. If every point of a closed set e on the interval ab is interior to one of a set of intervals I, then there exists a finite number of the intervals I with the same property.

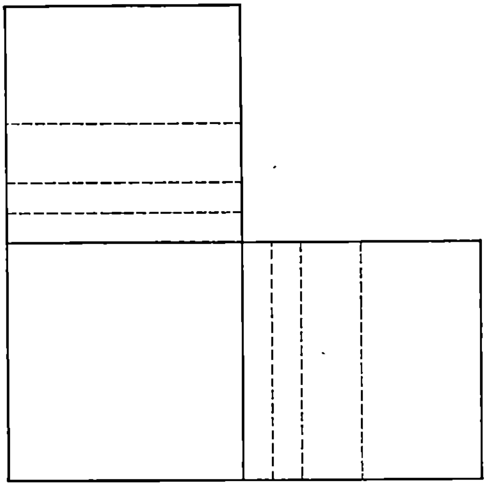

Fig. 3.

A closed set is one which contains all of its limit points. Every point ξ of ab is either in e, and therefore interior to an interval of I, or else is interior to an interval containing no point of e, since e is closed. When this remark is applied first at $\xi = a$, it is seen that there exists an interval ax such that all the points of e on ax are surely interior to a finite number of the intervals I. Let ξ next be the least upper bound of the values x defining such intervals. The same remark shows that if $\xi = b$ the lemma is true, while $\xi < b$ is impossible.

Consider now a function $P(x)$ which is continuous and monotonically increasing on the interval ab. If α is an interval with the end points $x_1 < x_2$, the notation $p(\alpha)$ will be used for the expression

$$(9) \qquad p(\alpha) = P(x_2) - P(x_1).$$

LEMMA 2. If the intervals α_k $(k = 1, 2, \cdots)$ are non-overlapping, and their points all among those of a second set β_k $(k = 1, 2, \cdots)$, then

$$\Sigma p(\alpha_k) \leqq \Sigma p(\beta_k).$$

The first series is surely convergent since the terms are all positive and the sum of the first m of them always less than $P(b) - P(a)$. If the second is finite then, since $P(x)$ is continuous, each β_k can be enlarged at each end to form an interval β_k' such that

$$p(\beta_k') < p(\beta_k) + \frac{\epsilon}{2^k}, \quad \Sigma p(\beta_k') < \Sigma p(\beta_k) + \epsilon.$$

The intervals $\alpha_1, \alpha_2, \cdots, \alpha_m$ form a closed set of points each interior to one of the intervals β'. Hence, by Lemma 1, these intervals are all enclosed in a finite set of intervals $\beta_1', \beta_2', \cdots, \beta_n'$, and it follows that

$$\sum_{i=1}^{m} p(\alpha_i) \leqq \sum_{k=1}^{n} p(\beta_k') < \sum_{k=1}^{\infty} p(\beta_k) + \epsilon.$$

Since this is true for every m and ϵ the inequality of the lemma must hold.

Point sets consisting of a denumerable set of intervals play an essential rôle in the definition of measure, as will be seen in the following pages. Let A and B always denote such sets, composed of the intervals α_k and β_k $(k = 1, 2, \cdots)$, respectively, the range of k being either finite or denumerably infinite.

LEMMA 3. A set of the type A is always expressible as a sum of a denumerable set of non-overlapping intervals.

For if the sum of the first n intervals $\alpha_1, \alpha_2, \cdots, \alpha_n$ of A is represented by a_n, it follows readily that

$$A = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots.$$

Furthermore each term in parenthesis can be expressed as a sum of a finite number of intervals not overlapping each other or the similar sums which precede.

The notation $p(A)$ will be used for the expression

$$p(A) = \sum_k p(\alpha_k),$$

where the intervals α_k are supposed to be non-overlapping. The number $p(A)$ so defined is uniquely determined by the set A. For Lemma 2 shows that

$$\Sigma p(\alpha_k) \leqq \Sigma p(\beta_k), \quad \Sigma p(\beta_k) \leqq \Sigma p(\alpha_k),$$

when the sums $\Sigma\alpha_k$ and $\Sigma\beta_k$ both consist of non-overlapping intervals and both define the same set of points A. Hence the value $p(A)$ is the same whatever set of non-overlapping intervals representing A is used to compute it.

THEOREM 2. *The totality of sets of the type A is closed under multiplication and addition, even for sums of a denumerable infinity of elements. The function $p(A)$ has the properties*

$$(10) \qquad p(A) + p(B) = p(A + B) + p(AB),$$

$$(11) \qquad p(A_1 + A_2 + \cdots) \leqq p(A_1) + p(A_2) + \cdots.$$

It is evident that the sum of a denumerable infinity of sets of the type A is a similar set, and the product AB also, since it is the sum of the product sets a_nb_n, where a_n and b_n are the sums of the first n intervals of A and B respectively. Furthermore the definition (9) of the function p for intervals shows that

$$p(a_n) + p(b_n) = p(a_n + b_n) + p(a_nb_n),$$

where the intervals of A, as well as those of B, may be supposed non-overlapping. But as n approaches infinity the terms of this equation approach as limits the corresponding terms of the equation (10), since $A + B$, for example, is the sum of the finite number of non-overlapping intervals constituting $a_n + b_n$, plus others which must be added to form $a_{n+1} + b_{n+1}$, and so on. The final conclusion of the theorem follows since every interval in the expression for $A_1 + A_2 + \cdots$, as a sum of non-overlapping intervals, occurs as a part or the whole of an interval in at least one of the sums A_k.

COROLLARY. *If A and B are entirely on the interval ab and include between them all of the points of ab, so that $A + B$ is the interval ab, then*

$$(12) \qquad p(A) + p(B) = P(b) - P(a) + p(AB).$$

This follows from the property (10) above and the fact that $p(ab)$ is by definition equal to the first two terms on the right.

§ 3. *The Measure Function $m(e)$ and Similar Functions.*

In the preceding section a monotonically increasing continuous function $P(x)$ gave rise to a function $p(e)$ which was well defined for all point sets consisting of a finite or denumerably infinite system of intervals. When $P(x) = x$ the value of $p(e)$ for one of these sets is called the measure $m(e)$ of e. The purpose of the present section is the extension of the definition of $p(e)$, or $m(e)$, to a larger class of point sets than those of the special type A.

An arbitrarily selected point set e can be enclosed in infinitely many ways in a set of intervals of the type A on ab. The greatest lower bound of the values $p(A)$ for such sets is called the *exterior value of p on e*,[*] or in the special case $P(x) = x$ the *exterior measure of e*,[†] and may be denoted by $\bar{p}(e)$. It follows immediately from this definition that

$$\begin{aligned}(13)\qquad &\bar{p}(e) \geqq 0, \quad \bar{p}(e_1) \geqq \bar{p}(e_2) \text{ when } e_1 \text{ contains } e_2,\\ &\bar{p}(e_1 + e_2 + \cdots) \leqq \bar{p}(e_1) + \bar{p}(e_2) + \cdots.\end{aligned}$$

The last of these properties is valid for a finite or denumerably infinite sum, and is a consequence of the second conclusion of Theorem 2.

The exterior value $\bar{p}(e)$ could equally well be defined as the greatest lower bound of the values $p(A)$ on sets A restricted to contain the points of e as interior points. For when the intervals of A are enlarged slightly, as in the proof of Lemma 2, so that $p(A)$ is only slightly increased, the result is a set containing e in its interior.

The value $\bar{p}(e)$ might be taken as the measure of the set e, in the special case $P(x) = x$, if it surely possessed the property of additivity which was essential in the existence proof of § 1, and which is moreover characteristic of all notions of measure. The requirement that the measure of the whole shall be the sum of the measures of its parts is a fundamental one. It turns out that the class of point sets possessing the special additive property of the following definition is a class in which as a whole the function $\bar{p}(e)$ is additive.

DEFINITION OF A MEASURABLE SET.[‡] *If the exterior values $\bar{p}(e)$ and $\bar{p}(Ce)$ satisfy the relation*

$$(14)\qquad \bar{p}(e) + \bar{p}(Ce) = P(b) - P(a),$$

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[*] VII, p. 101.

[†] VII, p. 22; V, vol. 1, 3d edition, p. 61; III, p. 104.

[‡] VII, pp. 22, 101; V, vol. 1, 3d edition, p. 62; III, p. 106.

then $p(e)$ is said to be well defined on e and its value is by definition

$$p(e) = \bar{p}(e) = P(b) - P(a) - \bar{p}(Ce).$$

When this happens for the special case $P(x) = x$ the set e is said to be MEASURABLE.

If a set of the type A encloses Ce then all points of the set $ab - A$ are interior to e. Lebesgue accordingly defines the interior measure of e in the case $P(x) = x$ to be

$$\underline{p}(e) = P(b) - P(a) - \bar{p}(Ce),$$

and calls the set e measurable if $\bar{p}(e) = \underline{p}(e)$. The definition of the preceding paragraph is clearly equivalent to this.

LEMMA 4. A necessary and sufficient condition that $p(e)$ be well defined on a set e is that there exist sets A, B enclosing e and Ce such that $p(AB) < \epsilon$.

The condition is clearly necessary. For suppose the relation (14) of the definition satisfied, and let A, B be chosen enclosing e, Ce, respectively, so that the first members of (12) and (14) differ by less than ϵ. Then a comparison of the second members shows that $p(AB) < \epsilon$. Conversely, suppose A and B can be chosen enclosing e and Ce for every ϵ so that $p(AB) < \epsilon$. Then the equation (12) shows that

$$\bar{p}(e) + \bar{p}(Ce) \leqq P(b) - P(a),$$

while the third of the relations (13) shows that

$$\bar{p}(e) + \bar{p}(Ce) \geqq \bar{p}(ab) = P(b) - P(a).$$

It is important to note, as a consequence of this lemma, that the value $p(A)$ of § 2 is precisely the value which would result from the definition last given. For from Lemma 2 and the fact that A encloses itself, it follows that $\bar{p}(A)$ is the $p(A)$ of § 2. Furthermore the portions B of the interval ab which are left after extracting the intervals $\alpha_1, \alpha_2, \cdots, \alpha_n$ enclose CA, and

$$p(AB) = \sum_{k=n+1}^{\infty} p(\alpha_k) < \epsilon$$

provided that n is sufficiently large, so that A is a set satisfying the requirements of Lemma 4.

The criterion of the lemma just proved is not always convenient of application. Another more useful one is that of the following theorem.

THEOREM 3. *Let A be a denumerable set of intervals enclosing e, and let e' be the part of A not in e, so that*

$$e + e' = A.$$

Then a necessary and sufficient condition that $p(e)$ be well defined is that for every $\epsilon > 0$ there exists a set A enclosing e with $\bar{p}(e') < \epsilon$. Under these circumstances the values $p(e)$, $p(A)$ satisfy the relation

$$p(e) \leqq p(A) < p(e) + \epsilon.$$

The condition of the theorem is necessary. For if A and B have been determined as in Lemma 4 so that $p(AB) < \epsilon$, then e' is in AB and will necessarily have $\bar{p}(e') < \epsilon$. To prove the sufficiency, suppose that a set A exists for which $\bar{p}(e') < \epsilon$. Then e' is enclosable in a set B_1 with $p(B_1) < \epsilon$. The rest of Ce exterior to A is enclosed in the intervals B_2 remaining in ab after $\alpha_1, \alpha_2, \cdots, \alpha_n$ have been extracted, and $p(AB_2) < \epsilon$ if n is sufficiently large. Hence Ce is enclosed in $B = B_1 + B_2$, and by (13)

$$p(AB) \leqq p(AB_1) + p(AB_2) < 2\epsilon.$$

A class $\mathcal{S}$ of point sets is said to be closed if it contains the complement of every one of its elements, and also the sum of every sequence $\{e_n\}$ whose elements are in $\mathcal{S}$. From the results of § 2 it follows at once that a closed class $\mathcal{S}$ is also closed under the operations of subtraction, multiplication of a finite or denumerably infinite set of elements, and taking complete or restricted limits of sequences.

A function $p(e)$ is said to be continuous if for every ϵ a δ can be found such that $p(e) < \epsilon$ for all sets e of diameter less than δ, and it is absolutely continuous if in this definition the word diameter is replaced by the word measure. It is said to be additive in $\mathcal{S}$ if

$$p(e_1 + e_2 + \cdots) = p(e_1) + p(e_2) + \cdots$$

for every finite or denumerably infinite set $\{e_n\}$ whose elements are distinct and in $\mathcal{S}$.

THEOREM 4. *The totality $\mathcal{S}$ of point sets e for which $p(e)$ is well defined according to the definition given above contains all intervals on ab and is closed. The function $p(e)$ itself is non-negative, continuous, and additive on $\mathcal{S}$. Furthermore for every set e of $\mathcal{S}$*

and every constant $\epsilon > 0$ *there exists a sum* A *of intervals enclosing* e *such that* $p(A - e) < \epsilon$.

It has been proved above that $p(e)$ is well defined on all denumerable sets of intervals A, and therefore also on every sub-interval of ab. Furthermore every element e_n of a sequence of sets in $\mathcal{S}$ is enclosable in a set A_n such that

$$(15) \qquad e_n + e_n' = A_n, \quad \bar{p}(e_n') < \frac{\epsilon}{2^n}.$$

Hence

$$(e_1 + e_2 + \cdots) + (e_1' + e_2' + \cdots) = (A_1 + A_2 + \cdots),$$

and from (13) and (15)

$$\bar{p}(e_1' + e_2' + \cdots) < \epsilon,$$

which proves that $p(e)$ is well defined on the sum of the elements e_n.

The continuity of $p(e)$ follows from the fact that $P(x)$ is continuous, and from the second of the relations (13), which says that $p(e) \leqq p(\alpha)$ whenever e is contained in the interval α.

To prove the additive property, consider first only two distinct sets e_1 and e_2 of $\mathcal{S}$. According to Theorem 3, they can be enclosed in sets A_1, A_2 in such a way that

$$e_1 + e_1' = A_1, \quad e_2 + e_2' = A_2,$$
$$\bar{p}(e_1') < \epsilon, \qquad \bar{p}(e_2') < \epsilon.$$

Then $p(e_1), p(e_2), p(e_1 + e_2)$ differ from $p(A_1), p(A_2), p(A_1 + A_2)$ by less than ϵ, ϵ, 2ϵ respectively. Furthermore, since e_1 and e_2 have no points in common,

$$p(A_1A_2) = p(e_1'(e_2 + e_2') + e_1e_2') < 2\epsilon,$$

$$(16) \quad p(A_1) + p(A_2) - p(A_1 + A_2) = p(A_1A_2) < 2\epsilon.$$

It follows that

$$(17) \qquad p(e_1) + p(e_2) - p(e_1 + e_2) = 0,$$

since the first member of this equation differs from the corresponding member of (16) by less than 4ϵ, and therefore from zero by less than 6ϵ, and ϵ is arbitrary.

The equation (17) implies the additive property for $p(e)$ for every finite number of sets e. When the sequence $\{e_n\}$ contains an infinity of elements, the series

$$p(e_1) + p(e_2) + \cdots$$

surely converges since

$$p(e_1) + \cdots + p(e_n) = p(e_1 + \cdots + e_n) \leqq p(ab).$$

From (17) also

$$p(e_1 + e_2 + \cdots) = p(e_1 + \cdots + e_n) + p(e_{n+1} + e_{n+2} + \cdots),$$

and with the help of (13)

$$p(e_{n+1} + e_{n+2} + \cdots) \leqq p(e_{n+1}) + p(e_{n+2}) + \cdots < \epsilon$$

if n is sufficiently large, which proves the desired property.

The last statement of the theorem follows at once from Theorem 3.

COROLLARY. *If e_2 is contained in e_1 and both are in $\mathcal{S}$, then*

$$p(e_1 - e_2) = p(e_1) - p(e_2).$$

If e is the limit of a sequence $\{e_n\}$ in $\mathcal{S}$, then

$$p(e) = \lim p(e_n).$$

The first conclusion is a consequence of the equation

$$e_1 = (e_1 - e_2) + e_2$$

and the additive property of $p(e)$. The second follows readily from the additive property when each e_n contains all the preceding, since then

$$e = e_1 + (e_2 - e_1) + (e_3 - e_2) + \cdots.$$

It is provable in a similar way when each e_n is contained in the preceding. For under that hypothesis

$$e = e_1 e_2 \cdots,$$

$$(18) \qquad Ce = Ce_1 + (Ce_2 - Ce_1) + (Ce_3 - Ce_2) + \cdots.$$

As a consequence of the inequalities

$$p(e_n e_{n+1} \cdots) \leqq p(e_n) \leqq p(e_n + e_{n+1} + \cdots)$$

the theorem follows for every limit e, since from the two special cases just considered and the relations (7) and (8)

$$p(e) = \lim p(e_n + e_{n+1} + \cdots) = \lim p(e_n e_{n+1} \cdots).$$

It is important to note that the limit of $p(e_n)$ is zero when each e_n contains the succeeding and all the sets together have no point in common. For then the set Ce in equation (18) is exactly the interval ab. The additive property applied to this equation shows that the limit of $p(Ce_n)$ is $p(ab)$, and the relation

$$p(e_n) + p(Ce_n) = p(ab)$$

shows the truth of the statement just made.

Every class $\mathcal{E}$ having the properties described in Theorem 4 contains all sets which are formed from intervals by addition and taking complements, or by repetition of these processes a finite or denumerably infinite number of times. These are the sets to which Borel* first extended the definition of measure, and are called measurable (B). They are important because they are necessarily contained in every class $\mathcal{E}$ which is closed and contains all intervals.

§ 4. *Measurable Functions.*

The definition of a measurable function given in § 1 is adequate for the purposes of the proof there given of the existence of the Lebesgue integral limit for a bounded function. It is unsatisfactory, however, for functions $f(x)$ which at some points have infinite values, because it takes no account of the abscissas x where this happens. It will be evident in the following pages that the application of Lebesgue integrals to non-bounded functions is a feature of the theory comparable in importance with the generalizations of the notions of measure and integration which have already been described, and it is highly desirable, therefore, to modify the definition of § 1 so that it may be more widely applicable. The definition given in this section is equivalent to the earlier one for bounded functions and has the necessary generality.

DEFINITION OF A MEASURABLE FUNCTION.† *Let e be a set of points x at each of which $f(x)$ has either a single finite value, or else one of the values $+\infty$ and $-\infty$. Then $f(x)$ is measurable on e if e is measurable, and if furthermore the set $e[f \geqq a]$ is measurable for every constant a.*

If a bounded function is measurable according to the definition of § 1, it is also measurable according to this definition.

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* I, p. 46.

† VII, p. 27; V, vol. 1, 3d edition, p. 69; III, p. 110.

For let M be greater than all the values of $f(x)$ on e, and let b be greater than M. The sets $e[f \geqq a]$ and $e[a \leqq f(x) < b]$ are the same, and the measurability of the latter implies that of the former. Conversely, the equation

$$e[a \leqq f < b] = e[a \leqq f] - e[b \leqq f]$$

shows that a function measurable under the definition of this section is also measurable according to that of § 1.

THEOREM 5. *If for a function $f(x)$ of the kind described above all the sets of any one of the four types*

$$e[f \geqq a], \quad e[f > a], \quad e[f < a], \quad e[f \leqq a]$$

are measurable for every a, then those of the other three have the same property, and f is measurable on e.

Suppose that the sets of the first type are all measurable. It follows that $e[f = a]$ is measurable for every a, since the sum of the sets $e[f \geqq a + 1/n]$ $(n = 1, 2, \cdots)$ is measurable, and $e[f = a]$ is the difference between $e[f \geqq a]$ and this sum. Hence the sets of the second type are also measurable. Furthermore the sum of the first and third sets is the measurable set e, and the fourth is the sum of the third and $e[f = a]$. In a similar manner the measurability of any one of the types implies the measurability of the other three.

COROLLARY. *If $f(x)$ is measurable on e then $e[f = a]$ and $e[a \leqq f < b]$ are measurable for all values of the constants a and b.*

LEMMA 5. If f and φ are both measurable on a set e then $e[f > \varphi]$ is measurable.

For there is always a rational number r between f and φ when $f > \varphi$, and the set $e[f > \varphi]$ is therefore the sum of the denumerable infinity of product sets $e[f > r] \cdot e[r > \varphi]$, where r is a rational number.

THEOREM 6. *If f and φ are both measurable on a set e, then each of the functions f, φ, $f + \varphi$, $f - \varphi$, $f\varphi$, $|f|$, and $1/f$ is measurable on every measurable sub-set of e where it is well-defined, and on the whole of e if an arbitrary constant value is assigned to it at all points where it is not defined.*

If a function f is measurable on e it is also measurable on every measurable sub-set e' of e, since $e'[f \geqq a]$ is the product of e' and $e[f \geqq a]$.

The points where $f \pm \varphi$ is not well-defined are those of the set where the values of f and $\pm \varphi$ are opposite infinities. This set is measurable since $e[f = +\infty]$ is the product of the measurable sets $e[f \geqq n]$ $(n = 1, 2, \cdots)$, and since $e[f = -\infty]$, $e[\varphi = \pm\infty]$ can be similarly shown to be measurable. On the measurable remainder e' of e it follows at once from the definition of a measurable function and Theorem 5 that the measurability of φ implies also that of $\varphi + c$ and $c\varphi$, where c is any constant. Hence the functions $f \pm \varphi$ are measurable on e' since the sets

$$e'[f \pm \varphi > a] = e'[f > a \mp \varphi]$$

are measurable for every a, by Lemma 5. Furthermore f^2 is measurable when f is so, since for every $a \geqq 0$

$$e[f^2 \geqq a] = e[f \geqq \sqrt{a}] + e[f \leqq -\sqrt{a}],$$

and the formula

$$4f\varphi = (f+\varphi)^2 - (f-\varphi)^2$$

shows that $f\varphi$ is measurable on the sub-set of e where f and φ and therefore $f\varphi$ are finite. On the remainder of e the product $f\varphi$ is definitely infinite on sets which are measurable, or else the product of an infinity by zero and indeterminate. The measurability of $|f|$ follows from the measurability of f^2 and the formula

$$e[|f| \geqq a] = e[\sqrt{f^2} \geqq a] = e[f^2 \geqq a^2],$$

which holds also for every $a \geqq 0$. Finally $1/f$ is measurable on the sub-set e' of e where $f \neq 0$, on account of the formulas

$$a > 0:\quad e'\left[\frac{1}{f} \geqq a\right] = e'\left[\frac{1}{a} \geqq f\right] \cdot e'[f > 0],$$

$$a = 0:\quad e'\left[\frac{1}{f} \geqq 0\right] = e'[f > 0],$$

$$a < 0:\quad e'\left[\frac{1}{f} \geqq a\right] = e'[f > 0] + e'\left[f \leqq \frac{1}{a}\right].$$

The last statement of the theorem follows since every set where one of the functions is indeterminate is measurable.

THEOREM 7.* *The upper bound $\varphi(x)$ of a set of functions*

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* VII, p. 29; VI, p. 42; V, vol. 1, 3d edition, p. 71.

$f_n(x)$ $(n = 1, 2, \cdots)$ *measurable on* e, *is measurable* *Similarly the upper and lower limits of a sequence of fun* $\{f_n\}$ *are measurable. If the sequence has a unique li* *limit is measurable.*

For the set $e[\varphi > a]$ is the sum of the measura $e[f_n > a]$ for the values $n = 1, 2, \cdots$. Further $\varphi_n(x)$ be the measurable upper bound of the set f_n, f The upper limit $L(x)$ of the sequence $\{f_n\}$ is by defini lower bound of the functions $\varphi_n(x)$, and hence also able. The sequence is said to have a unique limit, eq both the upper and the lower limit, when the upper and l limits are everywhere the same.

THEOREM 8.* *Let* $f(x, y)$ *be a function having a single rea value at each point of the region* $a \leqq x \leqq b$, $0 < y \leqq d$. *Suppose furthermore that* f *is continuous in* y *for every fixed* x, *and measurable in* x *for every fixed* y. *Then the functions*

$$\varphi(x) = \overline{\lim_{y=0}}\, f(x, y), \quad \psi(x) = \underline{\lim_{y=0}}\, f(x, y)$$

are measurable on ab.

Since $f(x, y)$ is continuous in y, the upper bound $\varphi_n(x)$ of the values $f(x, y)$ when x is fixed and $0 < y \leqq 1/n$, is the same as the upper bound of the denumerable set of values $f(x, r)$ where $y = r$ is a rational value of y on the interval $0 < y \leqq 1/n$. Hence by the first part of Theorem 7, $\varphi_n(x)$ is a measurable function of x. Further the function $\varphi(x)$ of the theorem is the limit of $\varphi_n(x)$ as n approaches infinity, and therefore measurable. Similar reasoning holds for $\psi(x)$.

§ 5. *Properties of Lebesgue Integrals of Bounded Functions.*

The integral of Lebesgue has properties similar to those of the classical integral of Riemann, but with two very important extensions. An integral of either type, regarded as a function $I(f, e)$ of the function f to be integrated and of the domain of integration e, has additive properties with respect to each argument similar to those described in the statements 2) and 4) of the theorem below. For the Riemann integral the domain of integration is usually an interval, and the additive property 2) is true for a finite number of them; whereas the additivity of the Lebesgue integral holds even when the domain of integration is decomposed into a de-

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[*] VII, p. 30; VI, p. 442.

numerable infinity of parts. The additive property with respect to the argument f is valid for a Riemann integral when f is the sum of a finite number or an infinite series of bounded integrable functions, provided that the series converges uniformly.* The only requirement prescribed below for this property of a Lebesgue integral is that f shall be the sum of a convergent series of measurable terms f_n for which the partial sums up to n terms have common bounds μ, M. It is one of the most remarkable characteristics of the integrals of Lebesgue that the requirement of uniformity of convergence is unnecessary for the establishment of this property. In § 6 some examples are given to illustrate these generalizations. The property 5) of the theorem is one which both integrals have in common, but the recognition of its validity for the Riemann integral was a consequence of the development of the newer theory. No proof of this property is given in this section because it is a special case of a similar theorem for summable functions which will be proved later.

THEOREM 9.† *The Lebesgue integrals of functions which are bounded and measurable have the following properties*:

1) *If f is bounded and measurable, with bounds μ, M on e, then*

$$\mu m(e) \leqq \int_e f dx \leqq M m(e).$$

The integral is well defined and absolutely continuous as a function $g(e)$ on the measurable sub-sets of e.

2) *If f is bounded and measurable on the sum e of a finite number or an infinite sequence of distinct measurable sets e_n, then*

$$\int_e f dx = \int_{e_1} f dx + \int_{e_2} f dx + \cdots.$$

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3) *If f and φ are bounded and measurable, and satisfy $f \leqq \varphi$ on e, then af and $|f|$ are also bounded and measurable on e and*

$$\int_e af dx = a \int_e f dx, \quad \int_e f dx \leqq \int_e \varphi dx, \quad \left| \int_e f dx \right| \leqq \int_e |f| \, dx.$$

4) *If each of a finite number of functions f_n is bounded and*

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* Hobson, Theory of Functions of a Real Variable, p. 540.

† VII, pp. 39–45; V, vol. 1, 3d edition, pp. 258–259; III, pp. 112–116.

measurable on e, then their sum has the same property and

$$\int_e f dx = \int_e f_1 dx + \int_e f_2 dx + \cdots.$$

The theorem remains true for the sum of a convergent series of functions f_n, provided that the partial sums up to n terms all have absolute values less than a constant M.

5) *If f is bounded and measurable on the interval ab, the function*

$$F(x) = \int_{ax} f dx$$

has f as derivative at all the points of ab except those of a set of measure zero.

The sum S of § 1 whose limit is the Lebesgue integral satisfies the inequalities

$$(\mu - \epsilon)m(e) \leqq S \leqq (M + \epsilon)m(e),$$

since every value η_k having $m(e_k) \neq 0$ lies between $\mu - \epsilon$ and $M + \epsilon$. Hence at the limit as ϵ approaches zero the inequalities of 1) are true. The absolute continuity of the integral as a function of e is an immediate consequence of this mean value theorem.

To prove the property 2) consider first the sum $e = e' + e''$ of two sets on each of which f is bounded and measurable. If S, S', S'' are sums for f on the sets e, e', e'' respectively, formed with the same ladder of values l_k, then $S = S' + S''$ and the additive property for two sets, or indeed a larger finite number of sets, is an easy consequence. The sum e of a sequence $\{e_n\}$ may be written in the form

$$e = e_1 + e_2 + \cdots + e_n + r_n,$$

where $m(r_n)$ approaches zero as n approaches infinity. Hence from the additive property for a finite number of sets

$$\int_e f dx = \int_{e_1} f dx + \cdots + \int_{e_n} f dx + \int_{r_n} f dx,$$

and with the help of the property 1) the last integral is seen to have the limit zero as n increases.

If a sum S is formed for f with values l_k, and a sum S' for af with values al_k, it is easily seen that $S' = aS$, so that the first of the properties 3) also holds when ϵ approaches zero. It is useful to note that a quite similar proof shows that

$$\int_e (a + f)dx = am(e) + \int_e f dx. \tag{19}$$

To prove the second part of 3), let e be decomposed into the sets $e_k = e[l_{k-1} \leqq f < l_k]$. Then by 2) and the mean value theorem of 1)

$$\int_e \varphi dx = \sum_{k=1}^{n} \int_{e_k} \varphi dx \geqq \sum_{k=1}^{n} l_{k-1} m(e_k).$$

But the last expression is a sum S for the function f, and the property desired is a consequence when ϵ approaches zero. The last part of 3) follows with the help of the first part when the result just obtained is applied to the pairs f, $\varphi = |f|$ and $-f$, $\varphi = |f|$.

The properties 2) and 3) with equation (19) justify the inequalities

$$\int_e (f + \varphi)dx \geqq \sum_{k=1}^{n} \int_{e_k} (l_{k-1} + \varphi)dx = \sum_{k=1}^{n} l_{k-1} m(e_k) + \int_e \varphi dx,$$

$$\int_e (f + \varphi)dx \leqq \sum_{k=1}^{n} \int_{e_k} (l_k + \varphi)dx = \sum_{k=1}^{n} l_k m(e_k) + \int_e \varphi dx,$$

from which it follows at the limit that

$$\int_e (f + \varphi)dx = \int_e f dx + \int_e \varphi dx.$$

This establishes the property 4) for every sum of a finite number of measurable functions. To prove that a similar property holds for series the following lemma is useful.

LEMMA 6. If a sequence $s_n(x)$ of functions measurable on e converges to a bounded limit f, then for every ϵ the sets $e_n = e[|f - s_n| > \epsilon]$ are such that $\lim_{n=\infty} m(e_n) = 0$.

The complete limit c of the sets e_n can contain no point, since a value x in c would be in an infinity of sets e_n, and therefore $|f - s_n|$ would exceed ϵ for an infinity of values of n. This is not possible if f is the limit of the sequence $\{f_n\}$ at x. It follows therefore from formula (8) and the corollary to Theorem 4 that

$$m(c) = \lim m(e_n + e_{n+1} + \cdots) = 0,$$

and hence that $\lim m(e_n) = 0$.

It is now possible to prove the additive property 4) for an infinite series of functions f_n. For let s_n be the sum of the first n terms of the series, and e_n the set described in the lemma. Then with the help of 3), 4), 2), and 1),

$$\left|\int_e f dx - \int_e s_n dx\right| \leqq \left|\int_{e-e_n} (f - s_n) dx\right| + \left|\int_{e_n} (f - s_n) dx\right|$$
$$\leqq \epsilon m(e) + 2Mm(e_n),$$

and this can be made arbitrarily small by taking ϵ sufficiently small and n sufficiently large.

COROLLARY. *If a sequence* $\{s_n\}$ *of functions measurable on* e *has a limit* f, *and all the elements of the sequence have the same bounds* μ, M, *then*

$$\lim_{n=\infty} \int_e s_n dx = \int_e f dx.$$

§ 6. *Examples of Measurable Sets and Functions.*

The definition of measure commonly accepted before the ideas of Borel and his followers were developed was the following well-known one of Jordan.* Let the interval ab be divided into a finite number of sub-intervals of norm δ, and let s, S be the sums of the intervals entirely in e, and of the intervals containing points of e but not necessarily entirely in e, respectively. Then s and S have limits as δ approaches zero which are called the interior and exterior measures σ, Σ of the set e. If the two are equal the set e is said to be measurable (J), and its Jordan measure is the common value $\sigma = \Sigma$.

It is provable that the totality $\mathfrak{F}$ of sets measurable (J), and the Jordan measure function, have properties similar to those described in Theorem 4, the principal and very important difference being that $\mathfrak{F}$ is additively closed, and the Jordan measure function is additive, for a finite number of sets only. A very simple example can be given to illustrate the greater power of the new definition of measure in this respect. A set consisting of a single point has measure zero according to either definition. It is provable directly from the definition, as will be indicated just below, or from the additive property of the class $\mathcal{B}$ described in Theorem 4, that the set of rational points on the interval $0 \leqq x \leqq 1$, which is denumerable, must have Borel-Lebesgue measure zero. But it is evident also that the Jordan measure of this set is not well defined, since the exterior measure of the set is 1 and the interior measure 0.

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* IX, p. 28.

It is reasonable to ask whether or not all sets have measure according to the definition of § 3; but it seems impossible to give a conclusive answer to the question at present. Sets not measurable have been constructed by Van Vleck and Lebesgue with the help of principles from the theory of aggregates which have been in dispute, and which so far remain unproved.

THEOREM 10. *Every denumerable set of points has measure zero according to the definition of measure of* § 3. *Further every closed set of points is measurable* (*B*).

If the elements of the denumerable set are denoted by x_n ($n = 1, 2, \cdots$), each x_n can be enclosed in an interval of length $\epsilon/2^n$, and the sum of the lengths of these intervals is ϵ. Consequently the measure of the set is zero.

A point which is exterior to a closed set e is enclosable in an interval containing no point of e in its interior, but whose end points are in e. The totality of such intervals is denumerable, since there is but a finite number of them with lengths greater than $1/n$. The complement of e is therefore this measurable set consisting of a denumerable set of intervals $\Sigma\alpha_n$, which implies that e is measurable (B) also.

The Riemann integral of a bounded function on an interval ab is a notion closely allied to that of Jordan measure. For let ab be divided into a finite number of intervals Δx_k ($k = 1, 2, \cdots, n$) of norm δ, and let m_k, M_k be the lower and upper bounds of the values of $f(x)$ in Δx_k. The two sums

$$s = \sum_{k=1}^{n} m_k \Delta x_k, \quad S = \sum_{k=1}^{n} M_k \Delta x_k$$

have limits σ, Σ as δ approaches zero, and when the two are equal the function $f(x)$ is said to be integrable (R) on ab, the value of the integral being the common limit $\sigma = \Sigma$. If $f(x) = 1$ at every point of a set e and $f(x) = 0$ elsewhere, the limits σ, Σ are exactly the Jordan interior and exterior measures of e, so that the existence of the integral of this particular function, and the measurability of e according to the Jordan definition, imply each other.

A necessary and sufficient condition that $f(x)$ be integrable (R) is that

$$\lim_{\delta=0} (S - s) = 0.\text{[*]} \tag{20}$$

Lebesgue[†] has transformed this condition in a most interesting and useful way, as indicated in the following theorem.

THEOREM 11. *A necessary and sufficient condition that a bounded function $f(x)$ be integrable in the sense of Riemann on ab, is that the set of discontinuities of $f(x)$ have measure zero.*

The proof of this theorem by Lebesgue made no use of the properties of Borel-Lebesgue measure except the concept of a set of measure zero. A proof of Theorem 12 below which has recently been given by de la Vallée Poussin[‡] makes use of reasoning which, with the help of the theorems of the preceding section, leads more directly to the desired result. Let $M(x)$ be the limit as δ approaches zero of the maximum of f on an interval of norm δ with x as interior point. Furthermore let $\Phi(x)$ be a function having the value M_k at interior points of each interval Δx_k of a partition of ab, and having an arbitrarily assigned value, say zero, at the division points of the partition. Then

$$S = \int_{ab} \Phi(x)dx,$$

where the integral is to be interpreted in the sense of Lebesgue. Consider now a sequence $\{S_n\}$ of sums with norms approaching zero, and let $\{\Phi_n\}$ be the corresponding sequence of functions Φ. Every element of this last sequence is measurable, and hence its upper and lower limits are measurable, by Theorem 7. Furthermore the totality of partition points of the sums S_n forms a denumerable set of measure zero, and at every other point the sequence $\{\Phi_n\}$ is easily seen to have the unique limit $M(x)$. It follows that $M(x)$ is also measurable, since it is identical with a measurable function except on a set of measure zero; and from the Corollary to Theorem 9

$$\int_{ab} M(x)dx = \lim_{n=\infty} \int_{ab} \Phi_n(x)dx = \lim_{\delta=0} S = \Sigma,$$

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[*] V, vol. 1, 3d edition, p. 255.
[†] III, pp. 29, 109.
[‡] VII, p. 55.

since the interval ab differs from the set of points in which $\lim \Phi_n = M(x)$ only by a set of measure zero. A precisely similar argument gives σ as the integral of the function $m(x)$ which is the limit as δ approaches zero of the minimum of f on an interval of length δ having x as interior point. Hence a necessary and sufficient condition for f to be integrable (R) is

$$\int_{ab} [M(x) - m(x)]dx = 0.$$

The integrand of this integral is everywhere positive or zero, and when the integral vanishes can exceed $1/n$ only on a set of points e_n of measure zero. Otherwise, by the property 1) of Theorem 9, the integral would be different from zero. Hence a necessary and sufficient condition that the integral shall vanish is that the set of discontinuities of f, which is the totality of points where $M(x) - m(x) > 0$ and identical with the sum of the sets e_n, should have measure zero.

THEOREM 12.[*] *A bounded function $f(x)$ which is integrable in the sense of Riemann on ab is also measurable on ab, and its Riemann and Lebesgue integrals are equal. In particular for a continuous function the sets $e[f \geqq \alpha]$ are measurable (B) for every α.*

The set of points $e[f \geqq \alpha]$ contains all of its limit points at which f is continuous. The other limit points are discontinuities of f and therefore form a set of measure zero, by Theorem 11. Hence $e[f \geqq \alpha]$ plus a set of measure zero is closed and therefore measurable, from which it follows that $e[f \geqq \alpha]$ itself is measurable. For the case of a continuous function this set is closed and therefore measurable (B). From the properties 1) and 2) of Theorem 9 the Lebesgue integral of f satisfies the inequalities

$$\sum_{k=1}^{n} m_k \Delta x_k \leqq \sum_{k=1}^{n} \int_{\Delta x_k} f dx \leqq \sum_{k=1}^{n} M_k \Delta x_k$$

for every partition of ab into intervals Δx_k of norm δ. But the first and last members of this inequality are the sums s, S whose limits as δ approaches zero are both equal to the Riemann integral of f. Hence the equality of the two integrals of f on ab is proved.

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[*] VII, p. 55; V, vol. 1, 3d edition, p. 259.

COROLLARY. *Every set e which is measurable in the sense of Jordan is also measurable according to the definition of* § 3, *and the two measures are equal.*

Let $f(x)$ be the function described above which has the value $f(x) = 1$ at all points of e, but which is zero elsewhere. If the Jordan measure of e is well defined it is equal to the Riemann integral of $f(x)$ on ab, and hence also equal to the Lebesgue integral. But the latter is the Borel-Lebesgue measure of e, so that the two measures are equal.

It may be of interest to consider for a moment a simple example of a function integrable according to Lebesgue, but not so according to Riemann. Let $f(x) = 2$ on the set e_1 of irrational points on the interval $0 \leqq x \leqq 1$, and let $f(x) = 1$ on the set e_2 where x is rational. This function has sums $s = 1$, $S = 2$ for every subdivision of the interval ab into sub-intervals Δx_k, and hence has no Riemann integral. On the other hand the Lebesgue integral has the well-defined value

$$\int_{01} fdx = \int_{e_1} fdx + \int_{e_2} fdx = 2.$$

THEOREM 13. *Let $\alpha(y)$ be the measure of the set $e[\mu \leqq f < y]$ for a function $f(x)$ measurable and with $\mu < f < M$ on e. Then $\alpha(y)$ is a monotonically increasing function and the Lebesgue integral of f is expressible in the forms*

$$\int_e fdx = \int_\mu^M yd\alpha = Mm(e) - \int_\mu^M \alpha(y)dy, \tag{21}$$

where the second integral is to be taken in the sense of Stieltjes described below, and the last is an integral of Riemann.

If $\varphi(y)$ and $\alpha(y)$ are two single-valued functions on the interval $\mu \leqq y \leqq M$, the corresponding Stieltjes integral limit is by definition

$$\int_\mu^M \varphi d\alpha = \lim_{\epsilon=0} \sum_{k=1}^{n} \varphi(\eta_k)[\alpha(l_k) - \alpha(l_{k-1})],$$

where the values l_k $(k = 0, 1, \cdots, n)$ with $l_0 = \mu$, $l_n = M$ define a partition of μM into sub-intervals of norm ϵ having $l_{k-1} \leqq l_k$, $l_{k-1} \leqq \eta_k \leqq l_k$. It is useful, as will be seen presently, to permit successive values l_{k-1}, l_k to be equal. The existence of the limit as thus defined is provable* for every pair of functions φ, α such that φ is continuous and α of limited variation on μM, and it is an immediate consequence of this definition and that of § 1 that the first of the equations (21) is true, since $m(e_k)$ in (3) is exactly the difference $\alpha(l_k) - \alpha(l_{k-1})$. The choice $l_0 = \mu$, $l_n = M$ for the sum in (3) is now permissible since by the hypothesis here used $\mu < f < M$ on e.

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* See, for example, Riesz, loc. cit., p. 37; Stieltjes, loc. cit., p. J. 71.

Suppose now that the Stieltjes integral exists for two functions φ, α. Then it also exists when the rôles of φ and α are interchanged, and furthermore*

$$\int_\mu^M \varphi d\alpha + \int_\mu^M \alpha d\varphi = \varphi(M)\alpha(M) - \varphi(\mu)\alpha(\mu). \tag{22}$$

For let the values η_k ($k = 1, 2, \cdots, n$; $\eta_1 = \mu$, $\eta_n = M$) define a partition of μM of norm $\epsilon/2$, and let l_k ($k = 1, 2, \cdots, n - 1$) be chosen for each sub-interval so that $\eta_k \leqq l_k \leqq \eta_{k+1}$. Then the values l_k with the addition of $l_0 = \mu$, $l_n = M$ form a partition of μM with norm ϵ and are such that $l_{k-1} \leqq l_k$, $l_{k-1} \leqq \eta_k \leqq l_k$. It is a matter of algebra only to show now that

$$\sum_{k=2}^{n} \alpha(l_{k-1})[\varphi(\eta_k) - \varphi(\eta_{k-1})] = -\sum_{k=1}^{n} \varphi(\eta_k)[\alpha(l_k) - \alpha(l_{k-1})] + \varphi(M)\alpha(M) - \varphi(\mu)\alpha(\mu).$$

Since by hypothesis the limit of the sum on the right exists, the same is true of that on the left, and the formula (22) is the limit of this when ϵ approaches zero. Furthermore formula (22) justifies easily the second of the expressions (21) when $\varphi = y$, since $\alpha(M) = m(e)$, $\alpha(\mu) = 0$.

THEOREM 14. *Every derivative number* Λ *of a function* $f(x)$ *continuous on* ab *is measurable, and the sets* $e[\Lambda \geqq \alpha]$ *are all measurable* (*B*).

If the definition of $f(x)$ is extended so that $f(x) = f(b)$ for $x > b$, the resulting function is continuous for $x \geqq a$ and its forward derivative numbers are by definition the upper and lower limits of the quotient

$$g(x, h) = \frac{f(x + h) - f(x)}{h} \tag{23}$$

as h approaches zero over positive values. The quotient is a continuous function of both x and h for $a \leqq x \leqq b, 0 < h$, and the fact that its limits are measurable (B) is a consequence of the last statement in Theorem 12, and the proofs of Theorems 7 and 8. A similar argument holds for the two backward derivatives.

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* Riesz, loc. cit., p. 37; Stieltjes, loc. cit., p. J. 72.

§ 7. *Summable Functions.*

It has been remarked in § 4 that a part of the effectiveness of the Lebesgue theory of integration is associated with its applications to functions which are not necessarily bounded. This is due to the fact that the upper and lower limits of every sequence $\{f_k\}$ of measurable functions are measurable, but they may have $+\infty$ or $-\infty$ as functional values as well as finite ones. Consider then a function f which is measurable according to the definition of § 4, and let it be decomposed in the form

$$f = \frac{|f| + f}{2} - \frac{|f| - f}{2} = \varphi - \psi, \tag{24}$$

with $\varphi = 0$ when $f = -\infty$ and $\psi = 0$ when $f = +\infty$. Each of the functions φ, ψ is $\geqq 0$, and measurable according to Theorem 6. Consider now the function φ_n defined by the conditions

$$\begin{aligned} \varphi_n &= \varphi \text{ when } \varphi \leqq n, \\ \varphi_n &= n \text{ when } \varphi > n. \end{aligned} \tag{25}$$

This function is also measurable, as one may readily verify.

DEFINITION OF A SUMMABLE FUNCTION AND ITS INTEGRAL.* *A function $\varphi(x) \geqq 0$ is summable on e if it is measurable on e and such that the limit*

$$\int_e \varphi dx = \lim_{n=+\infty} \int_e \varphi_n dx$$

exists. A function f of arbitrary sign is said to be summable on e if the functions φ and ψ of equation (24) are both summable on e. The integral of f is then defined to be

$$\int_e f dx = \int_e \varphi dx - \int_e \psi dx. \tag{26}$$

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[*] VII, p. 45; VI, p. 444; V, vol. 1, 3d edition, p. 260; III, p. 115.

It follows at once from this definition and the mean value theorem that if f is summable the set of values x where $f(x)$ is $+\infty$ or $-\infty$ has measure zero. When e is replaced by the interval ax the integral on the left in (26) becomes a function of x expressed as a difference of two increasing functions, and hence is of limited variation. Further properties of integrals of summable functions are summarized in the following theorem, the proof of property 5) being again postponed to § 9.

THEOREM 15. *If in the statement of Theorem* 9 *the words "bounded and measurable" are everywhere replaced by the word summable, the properties* 1)–5) *are still true, the only exception being that the additive property* 4) *does not necessarily hold for a denumerable infinity of functions.*

The proofs will be made first for functions which are nowhere negative. The law of the mean in 1) for summable functions is a consequence of the second formula in 3) which will be proved later. If $f(x)$ is summable on e it is so on every measurable subset of e, as one may see at once from the definition. The absolute continuity of the integral of a non-negative summable function follows, since for a sufficiently large value of n

$$\int_e \varphi dx \leqq \int_e \varphi_n dx + \frac{\epsilon}{2} \leqq nm(e) + \frac{\epsilon}{2},$$

and when $m(e)$ is sufficiently small this is less than ϵ.

If φ is non-negative and summable on a sum e of measurable sets e_k $(k = 1, 2, \cdots)$, then

$$\int_e \varphi_n dx \geqq \sum_{k=1}^{p} \int_{e_k} \varphi_n dx,$$

$$\int_e \varphi_n dx = \sum_{k=1}^{\infty} \int_{e_k} \varphi_n dx \leqq \sum_{k=1}^{\infty} \int_{e_k} \varphi dx.$$

By letting n and then p approach infinity in the first of these, and by letting n approach infinity in the second, one obtains the inequalities

$$\int_e \varphi dx \geqq \sum_{k=1}^{\infty} \int_{e_k} \varphi dx, \quad \int_e \varphi dx \leqq \sum_{k=1}^{\infty} \int_{e_k} \varphi dx,$$

which prove property 2).

The first and second relations under 3) follow readily from the definition of a summable function when it is noted that for non-negative functions and $a > 0$

$$(af)_{an} = af_n, \quad f_n \leqq \varphi_n. \tag{27}$$

The argument for $a < 0$ is similar. The third relation of 3) is evident for positive functions since then $|f| = f$.

The proof here given for the second of the properties 3) justifies more than is stated in the theorem. For if φ is positive and summable and $0 \leqq f \leqq \varphi$ it is a consequence of the second of the relations (27) that f must also be summable.

If the functions f, g are non-negative and $h = f + g$ one may verify the relation

$$h_n \leqq f_n + g_n \leqq h_{2n}.$$

The integration of this shows that when f and g are summable, the same is true of h, and at the limit when n approaches infinity

$$\int_e h dx \leqq \int_e f dx + \int_e g dx \leqq \int_e h dx.$$

Hence the additive property 4) holds for a finite number of non-negative functions.

For a function f of arbitrary sign the absolute continuity of 1), the additive property 2), and the first relation of 3), follow from the preceding proofs when f is decomposed as in equation (24). The property 4) can now be derived for the sum $h = f + g$ of two summable functions of arbitrary sign by subdividing e into sub-regions where no one of f, g, h changes sign. In each of these regions h is summable, since it does not change sign and is numerically less than or equal to a non-negative summable function, and the property 4) is a consequence of the proof for non-negative functions, provided that the functions in the relation $h = f + g$ are suitably transposed. Addition of these results shows that the same is true for the original region, since a function summable on each of a finite number of sets is readily seen to be summable on their sum, and since in that case the additive property 2) holds. The second of the relations 3) is a consequence of 4) and the first of the relations 3), since $\varphi - f \geqq 0$ and

$$\int_e \varphi dx - \int_e f dx = \int_e (\varphi - f) dx \geqq 0.$$

If f is decomposed as in equation (24) the value of $|f|$ is $\varphi + \psi$ which by 4) is summable and such that

$$\int_e |f|\, dx = \int_e \varphi dx + \int_e \psi dx \geqq \int_e \varphi dx - \int_e \psi dx = \int_e f dx.$$

One can also state conversely that $|f|$ summable implies that φ and ψ, and consequently f, are summable. Finally the mean value theorem is an immediate consequence of the second of the relations 3). In the statement of this theorem it is understood that one or both of the values μ, M may be infinite.

The notions of major and minor functions of a summable function $f(x)$ were devised by de la Vallée Poussin and described by him as being very precious in the theory of Lebesgue integrals. He has certainly applied them with great success.

DEFINITION OF MAJOR AND MINOR FUNCTIONS OF A SUMMABLE FUNCTION f.* *A function* $\Phi(x)$ *is a major function for a function* f *summable on* ab *if for some* $\epsilon > 0$ *it has the properties*

$$1) \qquad \int_{ax} f dx \leqq \Phi(x) - \Phi(a) < \int_{ax} f dx + \epsilon;$$

2) *every derivative number* Λ *of* Φ *satisfies the inequality* $\Lambda > f$ *at every value* x *where* $f(x) \neq +\infty$.

The same definition characterizes a minor function $\varphi(x)$ *if the signs before* ϵ *and* ∞ *are changed and the inequalities in* 1) *and* 2) *are inverted.*

The applicability of these functions is a consequence of the following theorem:

THEOREM 16. *There exists a major function* $\Phi(x)$ *for every function* $f(x)$ *summable on* ab *and every constant* $\epsilon > 0$. *A similar statement holds for minor functions.*

To prove this consider first a function $f(x)$ which is summable and non-negative on ab, with a ladder of values l_k $[k = 0, 1, \cdots; l_0 = 0, 0 < l_k - l_{k-1} < \epsilon/2(b-a)]$. Let the part e_k of ab where $l_{k-1} \leqq f < l_k$ be enclosed in the interior of a denumerable set A_k of non-overlapping intervals having $m(A_k - e_k) < \epsilon_k$, the constants ϵ_k being so chosen that $\Sigma l_k \epsilon_k < \epsilon/2$. If the portions of e_k and A_k on ax, and also their measures, are denoted by $e_k(x)$ and $A_k(x)$, respectively, then

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* V, vol. 1, 3d edition, p. 269; VII, p. 74; VI, p. 461.

$$\Phi(x) = \sum_k l_k A_k(x) \geqq \sum_k l_k e_k(x) \geqq \sum_k \int_{e_k(x)} f dx = \int_{ax} f dx.$$

Furthermore

$$\sum_k l_k A_k(x) - \sum_k l_k e_k(x) < \sum_k l_k \epsilon_k < \frac{\epsilon}{2},$$

$$\sum_k l_k e_k(x) - \int_{ax} f dx = \sum_k \int_{e_k(x)} (l_k - f) dx < \frac{\epsilon}{2},$$

so that by adding these two relations the function $\Phi(x)$ is seen to satisfy the condition 1) of the definition. But every value x where $f(x)$ is finite is in one of the sets e_k and therefore interior to the corresponding A_k. Consequently the term $l_k A_k(x)$ has derivative l_k at x, and since all the other terms of Φ are non-decreasing functions, it follows that all derivative numbers of Φ are at least equal to $l_k > f$.

For a function f of the form (24) a constant n can be selected so large that

$$\int_{ax} \varphi dx - \int_{ax} \psi_n dx - \int_{ax} f dx < \frac{\epsilon}{2}.$$

A major function for $\varphi - \psi_n$ with constant $\epsilon/2$ will therefore be a major function for f with constant ϵ. But $\varphi - \psi_n + n$ is non-negative and has a major function $\Phi(x)$ for $\epsilon/2$, and the function $\Phi(x) - nx$ plays the same rôle for $\varphi - \psi_n$.

A major function of $-f$ is the negative of a minor function for f, so that the theorem is proved in its entirety.

§ 8. *A Generalization of the Fundamental Theorem of Integral Calculus.*[*]

A function $G(x)$ of limited variation and continuous on ab may be defined as one expressible in the form

$$G(x) = P_1(x) - P_2(x), \tag{28}$$

where P_1 and P_2 are continuous and monotonically increasing. If the process of §§ 2, 3 is applied to P_1 and P_2, two classes of point sets $\mathcal{E}_1$, $\mathcal{E}_2$ and two functions $p_1(e)$, $p_2(e)$ are defined having the relationships described in Theorem 4. The greatest common subclass $\mathcal{E}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ contains all intervals on ab

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[*] VII, Chapitre VI; VI, § 8; IV.

and is closed. Furthermore the function

$$g(e) = p_1(e) - p_2(e)$$

is continuous and additive on $\mathcal{E}$, and such that for every set e of $\mathcal{E}$ and every constant $\epsilon > 0$ there exists a sum A of intervals enclosing e such that

$$p_1(A - e) + p_2(A - e) < \epsilon. \tag{29}$$

The functions $g(e)$ defined in this way are of a very general type which is the subject of the following theorems.

THEOREM 17. *Consider a function $g(e)$ which is continuous and additive on a closed class of point sets $\mathcal{E}$ containing all the intervals on ab. On the totality of subsets in $\mathcal{E}$ of an element e of $\mathcal{E}$ the function $g(e)$ has a positive maximum $p_1(e)$, and $g(e)$ is expressible as the difference of two non-negative functions*

$$g(e) = p_1(e) - [p_1(e) - g(e)] = p_1(e) - p_2(e).$$

The functions $p_1(e)$ and $p_2(e)$ are continuous and additive on $\mathcal{E}$. The sum

$$T(e) = p_1(e) + p_2(e)$$

is called the total variation of g.

Since $g(e)$ is continuous it is clear that its least upper bound on the subsets of a set e_1 must be positive or zero, if such a bound exists. If there were no such upper bound on e_1, one could select a subset e_2 so that $g(e_2)$ and $g(e_1 - e_2)$ are both numerically greater than unity, and so that g would also have no upper bound on e_2. By repeating this process a sequence $\{e_k\}$ would be found, with each element contained in the preceding, and having a greatest common subset e such that

$$e_1 = e + (e_1 - e_2) + (e_2 - e_3) + \cdots.$$

This would contradict the additivity of g, however, since each term after the first in the sum

$$g(e) + g(e_1 - e_2) + g(e_2 - e_3) + \cdots$$

would numerically be greater than unity, and its sum could not be $g(e_1)$.

The upper bound $p_1(e)$ is continuous since $g(e)$ is so, and it is also additive. For if the sets e_k are distinct and

$$e = e_1 + e_2 + \cdots,$$

one may cause g to vary on subsets of e so that it approaches $p_1(e)$. On the corresponding subsets of e_k the values of g will never exceed $p_1(e_k)$, and it follows from the additivity of g that

$$p_1(e) \leqq p_1(e_1) + p_1(e_2) + \cdots.$$

Furthermore if g varies on subsets of e_k so that its values on all the e_k's approach simultaneously the upper bounds $p_1(e_k)$, then it follows that

$$p_1(e) \geqq p_1(e_1) + p_1(e_2) + \cdots,$$

and the additivity of $p_1(e)$ is proved.

DEFINITION. *A class $\mathcal{E}$ of point sets is said to be a normal class for a function $g(e)$ if it has the properties*

1) *$\mathcal{E}$ is a closed class of measurable sets containing all the intervals on ab;*

2) *$g(e)$ is continuous and additive on $\mathcal{E}$;*

3) *for every element e of $\mathcal{E}$ and every constant $\epsilon > 0$ there exists a sum of intervals A enclosing e such that $T(A - e) < \epsilon$.*

A set A can be chosen satisfying condition 3) and at the same time having $m(A - e) < \epsilon$. For if A_1 satisfies 3), and A_2 is such that $m(A_2 - e) < \epsilon$, the product set $A = A_1A_2$ will have the two properties desired, since $T(e)$ and $m(e)$ are both positive functions. When the property 3) holds $g(e)$ is the limit of the values of g on sets e' enclosing e, in the sense that for every e and ϵ there exists a set A enclosing e such that

$$| g(e') - g(e) | \leqq | T(e') - T(e) | \leqq T(A - e) < \epsilon$$

for every e' in A and containing e. This is a consequence of the fact that $T(e)$ is non-negative and additive.

The function $p(e)$ of § 2 generated by a monotonically increasing continuous function $P(x)$ is an example of a function on a normal class $\mathcal{E}$, as one may see by extracting from the class $\mathcal{E}$ of Theorem 4 all except its measurable sets. The following theorem shows how the most general function $g(e)$ on a normal class is always expressible in terms of functions of this type.

THEOREM 18. *For every function $G(x)$ of limited variation the functions $P_1(x)$, $P_2(x)$ of equation (28) define as in §§ 2, 3 two functions $p_1(e)$, $p_2(e)$ with normal classes $\mathcal{E}_1$, $\mathcal{E}_2$. The* *totality $\mathcal{E}$ of measurable sets common to $\mathcal{E}_1$ and $\mathcal{E}_2$ is a normal class for the function*

$$g(e) = p_1(e) - p_2(e). \tag{30}$$

Conversely, every function $g(e)$ on a normal class $\mathcal{E}$ is identical on $\mathcal{E}$ with a function $g(e)$ generated in this way by a continuous function $G(x)$ of limited variation.

The first part of the theorem is a consequence of the remarks in the first paragraph of this section, except the statement that $\mathcal{E}$ is a normal class for $g(e)$. $\mathcal{E}$ is necessarily closed and contains all intervals, since it is the greatest common subclass of three classes with these properties. Furthermore, the property 3) of the definition follows at once from the inequality (29), since for the function (30) the total variation $T(e)$ is surely less than $p_1(e) + p_2(e)$.

If $g(e)$ has the normal class $\mathcal{E}$ then the function $p_1(e)$ of Theorem 17 is the lower bound of the values $p_1(A)$ on sets of intervals A enclosing e, on account of the property 3) of the definition of a normal class, and is identical with the values $\overline{p}_1(e)$ defined as in § 3 by the function

$$P_1(x) = p_1(\omega) \qquad (\omega = \text{the interval } ax).$$

The identity of p_1 and $\overline{p}_1$ on intervals is evident with the help of the additive property of p_1, and it persists for other sets e because $p_1(e)$ and $\overline{p}_1(e)$ are the lower bounds of p_1 and $\overline{p}_1$, respectively, on sets A of intervals containing e. Furthermore the relation (14) is true for every element of $\mathcal{E}$, since $\overline{p}_1 = p_1$ is additive, and it follows that $\mathcal{E}$ is necessarily contained in the totality of sets on which the exterior and interior values $\overline{p}_1(e)$, $\underline{p}_1(e)$ defined by $P_1(x)$ are identical. Similar remarks hold for $p_2(e)$, and the theorem is therefore proved.

DEFINITION. *The derivative numbers of a function $g(e)$ on a normal class $\mathcal{E}$ are the four derivative numbers of the function*

$$G(x) = g(\omega) \qquad (\omega = \text{the interval } ax). \tag{31}$$

The singularities of $g(e)$ are the points where all four of its derivatives are $+\infty$, or all four $-\infty$.

With these preliminary notions and theorems at hand it is possible to proceed to the proof of a series of lemmas which lead to the theorem which is the object of this section.

LEMMA 7. The derivative numbers of $g(e)$ are all summable on ab, and the set E of singularities of $g(e)$ has therefore measure zero. Futhermore it belongs to $\mathcal{E}$.

The derivative numbers of $g(e)$ are measurable, by Theorem 14, since $G(x)$ in equation (31) is continuous. The further proof can be made for the upper forward derivative Λ of the positive function $T(e)$ since the derivative numbers of $g(e)$ all have absolute values less than Λ. Let Λ_n be constructed from Λ as in (25), and let $\varphi(x)$ be one of the minor functions of Λ_n with constant ϵ. The function $T(\omega) - \varphi(x)$ has its upper forward derivative greater than or equal to zero, since every derivative number of φ is less than Λ. Consequently $T(ab) \geqq \varphi(b) - \varphi(a)$,* and by letting ϵ approach zero it follows that

$$T(ab) \geqq \int_{ab} \Lambda_n dx,$$

and Λ is seen to be summable. The set of points where $\Lambda = +\infty$ includes all the points of E, and has measure zero since Λ is summable.

Since the sets $e[\Lambda \geqq n]$, for every derivative number Λ of $g(e)$ and every positive integer n, are all measurable (B) by Theorem 14, and since their product is the set of points where all four derivative numbers of $g(e)$ are $+\infty$, it follows that E also is measurable (B) and belongs to $\mathcal{E}$.

LEMMA 8. On a set e where a derivative number Λ of $g(e)$ is never $-\infty$ it is true that

$$g(e) \geqq \int_e \Lambda dx. \tag{32}$$

The inequality can be inverted if $\Lambda \neq +\infty$ on e.

The proof will first be given for an interval ω with end points α, β. Let $\varphi(x)$ be a minor function with constant ϵ for the function Λ on the interval ω. Then the function

$$\psi(x) = g(\alpha x) - \varphi(x)$$

has its upper derivative $D\psi$, of the same type (forward or backward) as Λ, positive or zero, since

$$\Lambda \leqq D\varphi + D\psi, \quad \Lambda > D\varphi.$$

Hence $g(\omega) \geqq \varphi(\beta) - \varphi(\alpha)$, and the inequality of the lemma for $e = \omega$ is a consequence when ϵ approaches zero.

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* V, vol. 1, 3d edition, p. 99.

If a set of intervals A encloses in its interior all the points where $\Lambda = -\infty$ then the inequality (32) is true on $e = CA$. For in the first place the function

$$\psi(e) = g(e \cdot CA) + T(e \cdot A) \geqq g(e)$$

has $\mathcal{E}$ as a normal class. The conditions 1), 2) of the definition of a normal class are clearly satisfied, and 3) also holds since for every denumerable set of intervals B enclosing e the total variation T_ψ of ψ satisfies the relation

$$T_\psi(B - e) \leqq T([B - e] \cdot CA) + T([B - e] \cdot A) = T(B - e).$$

Furthermore the derivative number Λ_ψ of the same type as Λ is greater than or equal to Λ at every point x, and never takes the value $-\infty$, since this can happen to Λ only at points interior to A in the neighborhood of which $\psi(e) = T(e) \geqq 0$. Hence on every interval ω whatsoever

$$\psi(\omega) \geqq \int_\omega \Lambda_\psi dx \geqq \int_\omega \Lambda dx.$$

But this must be true also for every set e of $\mathcal{E}$, since the values of ψ and the last integral on such a set are limits of their values on sets B of intervals enclosing e, according to the remarks following the definition given above of a normal class. In particular for $e = CA$

$$\psi(CA) = g(CA) \geqq \int_{CA} \Lambda dx.$$

Finally the inequality (32) is true for every set e whatsoever on which $\Lambda \neq -\infty$, since a set A enclosing Ce encloses all points where $\Lambda = -\infty$, and

$$| g(CA) - g(e) | = | g(Ce) - g(A) | < \epsilon,$$

$$\left| \int_{CA} \Lambda dx - \int_e \Lambda dx \right| = \left| \int_{Ce} \Lambda dx - \int_A \Lambda dx \right| < \epsilon,$$

provided that A is chosen enclosing Ce in its interior and so that $g(A - Ce)$ and $m(A - Ce)$ are both sufficiently small.

Corollary 1. If a derivative Λ of $g(e)$ is finite valued at every point of e then

$$g(e) = \int_e \Lambda dx.$$

COROLLARY 2. The value of $g(e)$ is zero on every set e of measure zero on which a derivative number of g is everywhere finite valued, or on which two derivatives have everywhere the values $+\infty$ and $-\infty$, respectively.

The first corollary is true since on the set e there described the inequality (32) and its inverse are both justified by the lemma; and the first part of the second corollary follows at once since the integral vanishes when the measure of e is zero. For a set e satisfying the condition in the last part of Corollary 2 the inequality (32) holds with opposite senses for the two derivatives, and the integrals on the right are both zero.

THEOREM 19.* *For every derivative number* Λ *of a function* $g(e)$ *on a normal class* $\mathfrak{S}$*, the equation*

$$g(e) = g(eE) + \int_e \Lambda dx \tag{33}$$

holds, where e *is an arbitrary element of* $\mathfrak{S}$ *and* E *is the set of singularities of* g*. If a similar equation holds for a second set* E' *of measure zero and another function* Λ'*, then* $g(eE) = g(eE')$ *for every* e*, and* Λ *coincides with* Λ' *almost everywhere, i. e., except possibly on a set of measure zero.*

For let E_1 be the totality of points where Λ is infinite. Then

$$g(e) = g(eE_1) + g(eCE_1) = g(eE_1) + \int_e \Lambda dx$$

because of Corollary 1 to Lemma 8 and the fact that E_1 has measure zero. Furthermore $g(eE_1) = g(eE)$, by Corollary 2 to Lemma 8, since $E_1 - E$ is of measure zero and can be subdivided into sets on each of which either one of the derivatives of g is finite valued, or else two have everywhere the values $+\infty$ and $-\infty$, respectively. Hence the first part of the theorem is true.

If equation (33) holds for a second pair E', Λ' then

$$g(eE) + \int_e \Lambda dx = g(eE') + \int_e \Lambda' dx.$$

After replacing e by $eE + eE'$ it follows that the first two terms on the two sides are equal, and hence also the integrals, for every e. If the integrals are equal an application of the mean value theorem shows that the set of points where

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[*] VII, p. 93; VI, p. 476; V, vol. 1, 3d edition, p. 277.

$\Lambda - \Lambda' > 1/n$ must have measure zero for every positive integer n. The sum of these sets has therefore measure zero, and it includes all the points where $\Lambda - \Lambda'$ is positive. A similar argument holds for the points where $\Lambda - \Lambda'$ is negative.

COROLLARY 1. *The function g has all of its derivative numbers equal, and hence has a unique derivative, almost everywhere.*

For the equation (33) holds for all four of the derivative numbers of f, and it follows from the last statement of the theorem that they must be equal except on a set of measure zero.

COROLLARY 2. *The integral of a function $f(x)$ summable on ab,*

$$(34) \qquad g(e) = \int_e f(x)dx,$$

has $f(x)$ as its unique derivative almost everywhere.

For the function $g(e)$ is well defined on the normal class $\mathcal{S}$ including all measurable sets on ab. The equation (34) is of the same type as (33) and it follows that all four of the derivative numbers of $g(e)$ are equal to $f(x)$ almost everywhere. This corollary is the justification of the property 5) of Theorems 9 and 15, for the case when e is the interval ax.

COROLLARY 3. *A necessary and sufficient condition that $g(e)$ shall be the integral (34) of a function $f(x)$ summable on ab, is that $g(e)$ be additive and absolutely continuous on the normal class on which it is defined.*

The condition is clearly necessary since by Theorem 15 every integral of a function summable on ab is additive and absolutely continuous on the totality $\mathcal{S}$ of measurable sets on ab. On the other hand such a function on a normal class $\mathcal{S}$ is expressible in the form (33), with $g(eE)$ vanishing because g is absolutely continuous and eE has measure zero.

COROLLARY 4. *The function $g(eE)$ is called the function of singularities of g. If E' is a set of measure zero including E then $g(eE) = g(eE')$ for every e. If A is a denumerable set of intervals enclosing E then*

$$(35) \qquad g(eE) = \lim_{mA=0} g(eA).$$

If E_1 and E_2 are the parts of E on which all four derivative numbers of g are $+\infty$, $-\infty$, respectively, then

$$(36) \qquad g(eE) = g(eE_1) + g(eE_2)$$

and the functions on the right satisfy the inequalities

$$g(eE_1) \geqq 0, \quad g(eE_2) \leqq 0. \tag{37}$$

Each of the functions in (36) *has a unique derivative equal to zero almost everywhere, and each vanishes if its argument contains no perfect subset.*

To prove the first statement of the corollary one has only to substitute eE' for e in equation (33). A similar substitution of eA for e justifies the formula (35). The inequalities (37) are immediate consequences of Lemma 8. The fact that $g(eE)$ has derivative zero almost everywhere is evident because $g(e)$ and the integral in equation (33) both have derivatives equal to Λ almost everywhere. If the expression (36) is substituted in equation (33) it follows that $g(eE_2)$ is the function of singularities of $g(e) - g(eE_1)$, and hence must also have derivative zero almost everywhere. A similar argument holds for $g(eE_1)$.

The complement of eE can be enclosed in the interior of a denumerable set of intervals A in such a way that

$$| g(eE) - g(CA) | = | g(A) - g(CeE) | < \epsilon. \tag{38}$$

If the intervals of A are thought of as not including their end points, the set CA is necessarily closed* and the sum of a denumerable set and a perfect set.† But CA is interior to eE and consequently denumerable if eE contains no perfect subset. It follows that $g(CA)$ is zero, since g vanishes on a set consisting of a single point, and CA is the sum of a denumerable number of such sets. The inequality (38) now shows that $g(eE)$ is zero since it must be true for every ϵ. Similar arguments are applicable for $g(eE_1)$ and $g(eE_2)$.

It is a consequence of the Lemma 8, or also of the formulas (33) and (37), that $g(e)$ is necessarily positive or zero on a set e where a derivative number Λ of g is non-negative. If $\Lambda > 0$ on a subset of e of measure greater than zero, then $g(e) > 0$. Consequently on an arbitrary set e the function $g(e)$ attains its greatest value on the subset of e where $\Lambda \geqq 0$, and it follows readily that the total variation of $g(e)$ is given by the formula

$$T(e) = g(eE_1) - g(eE_2) + \int_e | \Lambda | \, dx. \tag{39}$$

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* VII, vol. 1, 3d edition, p. 54.

† Ibid., p. 52.

§ 9. *Applications of the Preceding Theorems to Functions $G(x)$, and an Example.*

The results described in the preceding section justify a series of notable properties of a function $G(x)$ of limited variation on an interval ab, and the formula (33) in particular interpreted for $G(x)$, gives extensions of the well-known fundamental theorem of the integral calculus which have many important applications.

In the first place every function $G(x)$ of limited variation on ab satisfies the relation

$$G(x) - G(a) = g(\omega) \qquad (\omega = \text{the interval } ax)$$

with a function $g(e)$ as in § 8 (Theorem 18), and has a unique derivative $F(x)$ almost everywhere (Theorem 19, Corollary 1). If arbitrary finite values are assigned to $F(x)$ at the set of points of measure zero where the derivative of $G(x)$ is not defined, then $F(x)$ is summable (Lemma 7). The variation $V(x)$ of $G(x)$ on the portion between a and x of the set E of singularities where $F(x)$ is $+\infty$ or $-\infty$, is defined to be the limit

$$\lim_{mA=0} \sum_{\alpha} \Delta_\alpha G = \lim_{mA=0} g(A), \tag{40}$$

where A is a denumerable set of non-overlapping intervals α enclosing the part of E on ax, and where for an interval α with end points x_1 and x_2

$$\Delta_\alpha G = G(x_2) - G(x_1).$$

This limit surely exists (Theorem 19, Corollary 4), and from formula (33)

$$G(x) - G(a) = V(x) + \int_{ax} F(x)dx. \tag{41}$$

The customary theorem of the integral calculus is that for every function $G(x)$ which has a bounded derivative $F(x)$ integrable on the interval ab in the sense of Riemann

$$G(x) - G(a) = \int_{ax} F(x)dx. \tag{42}$$

The formula (41) is a generalization of this for functions of limited variation, and has precisely the form (42) when the set of singularities E contains no perfect subset (Theorem 19, Corollary 4), in particular when a derivative number of $G(x)$ is everywhere finite valued. It is provable furthermore that every function $G(x)$ which has a derivative number $F(x)$ finite valued and summable on ab, is necessarily of limited variation on ab, and therefore satisfies formula (42). For let $\Phi(x)$ be a major function for $| F(x)|$. Then each of $\Phi(x)$ and $\Phi(x) - G(x)$ has a derivative number positive or zero, and is necessarily increasing on ab. Consequently

$$G(x) = \Phi(x) - [\Phi(x) - G(x)]$$

is the difference of two monotonically increasing functions.

A function $G(x)$ is said to be absolutely continuous on ab if the limit (40) is zero for all sets A irrespective of their relationship to E. A necessary and sufficient condition that $G(x)$ shall be expressible in the form (42) in terms of a summable function $G(x)$, is that $G(x)$ be absolutely continuous on ab (Theorem 19, Corollary 3). An integral of a summable function of the form in (42) is necessarily of limited variation, as one may see by replacing e by the interval ax in the formula (26), and its total variation on the interval ab is

$$T = \int_{ab} | F| \, dx,$$

according to formula (39). Furthermore such an integral has $F(x)$ as its derivative almost everywhere (Theorem 19, Corollary 2). If $G(x)$ is absolutely continuous and has a derivative number zero almost everywhere, then $G(x)$ must be constant, as one sees from formula (42).

The problem of determining whether or not a given function $F(x)$ is a derivative or a derivative number of a continuous function $G(x)$ was one of those which Lebesgue first considered. If $F(x)$ is continuous the function $G(x)$ defined by equation (42) is always a continuous anti-derivative of $F(x)$. But if $F(x)$ is discontinuous it may not be a derivative number of any continuous function whatsoever. A very simple example suffices to show this. Let $F(0) = 1$ and $F(x) = 0$ for $0 < x \leqq 1$. This function is bounded and measurable, and a function $G(x)$ which has $F(x)$ as a derivative number must therefore be expressible in the form (42). The function $G(x)$ so defined is clearly a constant and has everywhere the unique derivative zero. The method of procedure suggested by this example is applicable in general to finite valued summable functions. If such a function $F(x)$ is a derivative number of $G(x)$, then the latter must be expressible in the form (42). It has $F(x)$ as derivative almost everywhere, and it may be possible by actual test to determine whether or not this relationship is universally valid.

It may be of interest to consider an example of a function $G(x)$ of limited variation on the interval 01, for which the variation V in formula (41) is not zero.* Let $G(0) = 0$, $G(1) = 1$, and $G(x) = 1/2$ on the interval $1/3 < x < 2/3$. On the middle thirds of the two remaining intervals $G(x)$ is to have the values 1/4, 3/4 respectively. The process may be continued indefinitely. After division of 01 into 3^n equal parts the sum of the intervals on which $G(x)$ is not defined is $(2/3)^n$, so that the total set Ξ where G is not defined by this process has measure zero. Every point ξ of Ξ is therefore a limit point of points x of the set X where G has already been specified, and it is not difficult to show that $G(x)$ has a definite limit as x approaches ξ on the set X. This limit is to be taken as the value $G(\xi)$. The function $G(x)$ so determined on the whole interval 01 is monotonically increasing, and has derivative zero everywhere except at the points of the set Ξ of measure zero. Hence in formula (41) the integral vanishes and

$$1 = G(1) - G(0) = V(1).$$

It should be emphasized here in closing that the properties of functions of limited variation described in the paragraphs just above are a small part only of the applications of Lebesgue integrals. Every discussion involving definite integrals may give rise to important new results if the integrals are reinterpreted in the sense of Lebesgue, If an arc in space is rectifiable the functions $x(t)$, $y(t)$, $z(t)$ $(t_1 \leqq t \leqq t_2)$ defining it are of limited variation and have derivatives almost everywhere. The length of the arc is the value of the integral

$$\int_e \sqrt{x'^2 + y'^2 + z'^2}\, dt$$

taken in the Lebesgue sense over the set e on which the three derivatives exist simultaneously.† Vallée Poussin has enumerated the various hypotheses under which the formula

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* I am indebted to Mr. K. W. Lamson for the suggestion that a function of this type would illustrate the formula. See also III, p. 13.

† III, p. 125.

$$\int_e f dx = \lim_{n=\infty} \int_e s_n dx$$

holds when $\lim s_n = f$.* The conditions described in the Corollary to Theorem 9 can be greatly generalized to cover cases in which the functions s_n are not necessarily bounded in in their totality. The formulas for integration by parts, and for transformation of simple and double integrals, have new and more powerful interpretations in the Lebesgue theory,† and a multiple integral is reduced to an iterated integral by a particularly beautiful theorem.‡ If a function $f(x, y)$ is measurable and bounded on a two-dimensional set E then

$$\int\int_E f(x, y) dx dy = \int_X \int_{Y_x} f(x, y) dy dx$$

where X is the projection of E on the x-axis, and Y_x is the section of E on the ordinate over the abscissa x. The inner integral on the right does not exist for every x in the projection X, but it turns out that this does not affect its integrability over X since the set of points where it is not well-defined has measure zero. The problem of determining a function whose Fourier constants are given is really the problem of the summation of a Fourier series which may or may not be convergent according to the usual definitions. Riesz and Fischer§ with the help of the Lebesgue theory proved the existence of a solution under very general circumstances which were later still further generalized by Riesz.‖ The theory of Fourier series has been retouched in many other places also as a result of contact with the new integrals.¶ But it is impossible to list here in detail the rapidly increasing number of applications. More important than any one of them by itself are the new habits of thinking of and dealing with discontinuities too serious to be handled by the older forms of integration, and these will be the permanent legacies of the theory of Lebesgue.

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* VI, p. 443; V, vol. 1, 3d edition, p. 263; VII, p. 48.

† VI, pp. 465, 500; V, vol. 1, 3d edition, pp. 279 ff.

‡ VII, p. 50; V, vol. 2, 2d edition, p. 120.

§ See, for example, Lalesco, Introduction à la théorie des équations intégrales, p. 95.

‖ Loc. cit.

¶ See Lebesgue, Leçons sur les séries trigonométriques; and V, vol. 2, 2d edition, pp. 130 ff.

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II. Lebesgue, "Intégrale, longeur, aire," *Annali di Matematica Pura ed Applicata*, Serie 3, vol. 7 (1902), p. 231.

III. Lebesgue, Leçons sur l'Intégration.

IV. Lebesgue, "Sur l'intégration des fonctions discontinues," *Annales Scientifiques de l'Ecole Normale Supérieure*, vol. 27 (1910), p. 361.

V. Vallée Poussin, Cours d'Analyse Infinitésimale, vol. 1, 2d and 3d editions; vol. 2, 2d edition.

VI. Vallée Poussin, "Sur l'Intégrale de Lebesgue," *Transactions of the American Mathematical Society*, vol. 16 (1915), p. 435.

VII. Vallée Poussin, Intégrales de Lebesgue.

VIII. Radon, "Theorie und Anwendungen der absolut additiven Mengenfunktionen," *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften zu Wien*, vol. 122 (1913), Abtheilung IIa, p. 1295.

IX. Jordan, Cours d'Analyse, vol. 1, 2d edition.

NOTES.

THE July number (volume 18, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Set of independent postulates for betweenness," by E. V. HUNTINGTON and J. R. KLINE; "Haskins's momental theorem and its connection with Stieltjes's problem of moments," by E. B. VAN VLECK; "Point sets and allied Cremona groups (part III)," by A. B. COBLE; "On the second derivatives of the extremal integral for the integral $\int F(y; y')dt$," by ARNOLD DRESDEN; "Concerning singular transformations B_k of surfaces applicable to quadrics," by LUIGI BIANCHI; "Types of (2, 2) point correspondences between two planes," by F. R. SHARPE and VIRGIL SNYDER.

THE July number (volume 39, number 3) of the *American Journal of Mathematics* contains: "A concomitant curve of the plane quartic," by TERESA COHEN; "On two related transformations of space curves," by W. C. GRAUSTEIN; "The space problem of the calculus of variations in terms of angle," by P. R. RIDER; "Derivation of the complementary theorem from the Riemann-Roch theorem," by S. BEATTY; "On the equivalence of relations $K_{q_1 q_2 m}$," by E. W. CHITTENDEN; "Some properties of certain finite algebras," by EDWARD KIRCHER; "The primitive groups of class 15," by W. A. MANNING;

"Multiple integrals over infinite fields and the Fourier multiple integral," by B. H. CAMP.

NOTICE has been sent to the members of the Circolo matematico di Palermo that the society has temporarily suspended publication of the *Rendiconti*.

THE spring meeting of the Swiss mathematical society, which was held at Zurich on May 30 under the presidency of MARCEL GROSSMANN, was devoted to hearing an address by Professor J. HADAMARD, of Paris, on "The notion of analytic function and the partial differential equations."

AT the meeting of the Edinburgh mathematical society held on June 8 the following papers were read: By E. T. WHITTAKER, "On the expression of compound determinants as simple determinants"; by W. L. MARR, "A closed system of triangles apolar to a given triangle, and derivative systems"; by J. F. TINTO, (a) "On a group of transformations connected with the 27 lines of the cubic surface"; (b) "On the plane representation of a certain class of homaloidal surfaces."

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1917–1918:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mathematical theory of fluids, three hours.—By Professor L. DONATI: Thermodynamics and thermokinetics; Electrodynamics and radiations, three hours.—By Professor F. ENRIQUES: Geometrical theory of algebraic equations and functions, three hours.—By Professor S. PINCHERLE: Theory of analytic functions; the different points of view in the theory of linear differential equations, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Theory of numbers in rational and in any quadratic field; classical topics in asymptotic arithmetic, four hours.—By Professor E. DANIELE: Vibrations; applications to optics, four hours.—By Professor G. SCORZA: Geometry on an algebraic curve and abelian integrals, especially the reducible ones, three hours.—By Professor C. SEVERIN: Theory of integral equations, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: (at present with the army).—By Professor G. LORIA: n-dimensional geometry, three hours.—By Professor O. TEDONE: Optics; Phenomena of interferences and diffraction, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics: Newton and Leibniz, three hours.—By Professor P. DEL PEZZO: Cremona's transformations between planes and spaces, with applications to the analysis of singularities of curves and surfaces and to the plane representation of surfaces, three hours.—By Professor A. DEL RE: Analysis of Grassmann with applications; especially n-dimensional vector analysis, four and one half hours.—By Professor R. MARCOLONGO: Hydrodynamics, three hours.—By Professor D. MONTESANO: Birational transformations of space; Kantor's transformations; rational surfaces of the fourth and fifth orders, three hours.—By Professor E. PASCAL: Selected chapters of analysis, three hours.—By Professor L. PINTO: Electrostatics and magnetism with particular regard to the method of images and to the theory of dielectrica, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Harmonic and polyharmonic functions; topics in partial differential equations, four hours.—By Professor A. COMESSATTI: Introduction to algebraic geometry, three hours.—By Professor T. LEVI-CIVITA: Hydrodynamics, four hours.—By Professor G. RICCI: Absolute differential calculus with applications to the theory of elasticity, four hours.—By Professor F. SEVERI: Differential geometry, four hours.—By Professor A. TONOLO: Partial differential equations of the second order, three hours.—By Professor ————: Non-archimedean geometry, three hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Differential equations of the first order and calculus of variations, three hours.—By Professor M. DE FRANCHIS: Noneuclidean geometry and introduction to differential geometry, three hours.—By Professor M. GEBBIA: Electricity and magnetism (advanced part), four and one half hours.—By Professor A. SIGNORINI: Theory of elasticity, four hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Geometry of the hyperspaces, three hours.—By Professor U. CISOTTI: Electrodynamics, three hours.—By Professor F.

GERBALDI: Functions of a complex variable; elliptic functions, three hours.—By Professor G. VIVANTI: Integral equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Projective geometry of hyperspaces, three hours.—By Professor L. BIANCHI: (I) Existence theorems in the theory of ordinary and partial differential equations. (II) Applications to the infinitesimal geometry of curves and surfaces, four and one half hours.—By Professor U. DINI: Series, with particular regard to the divergent ones and to the various meanings of the sum of a series, four and one half hours.—By Professor G. H. MAGGI: Selected topics in the dynamics of continuous systems, four and one half hours.—By Professor P. PIZZETTI: Introduction to spherical astronomy; determination of an elliptic orbit; interpolation; variation of arbitrary constants and theory of perturbations.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical and kinematical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Algebraic plane and twisted curves, three hours.—By Professor U. CRUDELI: Arithmetic theory of binary and ternary forms, three hours. —By Professor L. SILLA: Differential equations of dynamics, three hours. By Professor V. VOLTERRA: Theory of the rotation of bodies possessing polycyclic interior motions, and the problem of the variation of latitudes, three hours.—Thermodynamics with applications; theory of explosives, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Hydrodynamics, three hours.—By Professor G. FUBINI: Abelian, elliptic and modular functions, three hours.—By Professor C. SEGRE: Applications of abelian integrals to geometry, three hours.—By Professor C. SOMIGLIANA: Elasticity and optics, three hours.

PROFESSORS U. CISOTTI and A. VITERBI, of the University of Pavia, have been elected corresponding members of the Reale Istituto Lombardo.

PROFESSORS U. CISOTTI, of the University of Pavia, and E. DANIELE, of the University of Catania, have been promoted to full professorships of mathematical physics.

DR. V. STRAZZERI has been appointed instructor in analytic and projective geometry at the University of Palermo.

THE De Morgan medal has been awarded by the London mathematical society to Professor W. H. YOUNG. The award is made every three years. The last three recipients were Glaisher (1908), Lamb (1911), and Larmor (1914).

PROFESSOR C. A. WALDO, of Washington University, St. Louis, has retired from active service.

PROFESSOR L. E. DICKSON, of the University of Chicago, is spending the present academic half-year as visiting professor at the University of California.

DR. H. N. WRIGHT, of the University of California, has been appointed professor of mathematics and dean of the faculty at Whittier College.

PROFESSOR J. L. RILEY, of the state normal school at Talequah, Okla., has been appointed professor of mathematics in the Junior Agricultural and Mechanical College of Texas, at Stephenville, Tex.

PROFESSOR W. A. GARRISON, of Union College, has been appointed professor of mathematics in King College, Bristol, Tenn.-Va.

MR. F. S. NOWLAN has been appointed professor of mathematics at Brandon College, Brandon, Manitoba.

AT Hamilton College associate professor W. M. CARRUTH has been promoted to a full professorship of mathematics.

AT the University of Cincinnati assistant professors LOUIS BRAND and C. N. MOORE have been promoted to associate professorships of mathematics. Mr. J. H. KINDLE and Dr. E. S. SMITH have been promoted to assistant professorships of mathematics.

AT the University of Kansas assistant professor E. B. STOUFFER has been promoted to an associate professorship of mathematics.

AT Dartmouth College Drs. L. C. MATHEWSON and C. R. DINES have been promoted to assistant professorships of mathematics.

DR. F. B. HITCHCOCK, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

MR. H. HALPERIN, of Vanderbilt University, has been appointed assistant professor of mathematics at the University of Arkansas.

PROFESSOR H. C. GOSSARD, of the University of Oklahoma, has been appointed to an instructorship in mathematics in the U. S. Naval Academy.

MR. K. W. LAMSON has been appointed instructor in mathematics at Columbia University.

MR. T. N. SIMPSON, of Converse College, Spartanburg, South Carolina, has been appointed instructor in mathematics at the University of Texas.

PROFESSOR GIUSEPPE VERONESE, of the University of Padua, died July 17, 1917, at the age of 63 years. He had been professor of mathematics at Padua since 1883, was a member of the Accademia dei Lincei, of the national Italian academy (the forty) and of many other learned societies, and was the author of well-known works on hypergeometry.

PROFESSOR G. W. HARTWELL, of Hamline University, died July 23, at the age of thirty-six years. Professor Hartwell had been a member of the American Mathematical Society since 1907.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

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Slichter (C. S.). See Dowling (L. W.).

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LIBRARY.

THE TWENTY-FOURTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

At the invitation of Adelbert College and the Case School of Applied Science, Cleveland, Ohio, the twenty-fourth summer meeting of the Society was held at these institutions on Tuesday, Wednesday, and Thursday, September 4–6, 1917. This was the Society's second visit to Cleveland, the annual meeting having been held there in the winter of 1912–1913. On the present occasion the interest was reinforced by the meeting of the Mathematical Association of America, immediately following on September 6–7. The arrangements, which were in charge of a committee representing both organizations, included a joint session on Thursday morning, at which Professor L. P. Eisenhart presented an address on "Darboux's Contribution to Geometry," and a joint dinner on Wednesday evening, attended by seventy-six members and friends, to whom President Thwing, of Western Reserve University, spoke a word of greeting, which was followed by a number of informal responses to the calls of the toast-master, Professor Huntington. The programme on Wednesday afternoon included an inspection of the harmonic analysis apparatus of Professor Miller, of the Case School, and an organ recital in the chapel. On Thursday afternoon President Thwing gave a garden party in honor of the visiting societies. Luncheon was served on each day at the Case Club, whose building was thrown open to the members afternoons and evenings. At the close of the meeting a vote of thanks was tendered to the authorities of the two colleges for their generous hospitality.

The meeting included the usual morning and afternoon sessions on Tuesday and Wednesday and the joint session on Thursday morning. The following sixty-four members were in attendance:

Professor O. P. Akers, Dr. Florence E. Allen, Professor R. B. Allen, Professor L. D. Ames, Professor Frederick Anderegg, Professor R. C. Archibald, Professor G. N. Armstrong, Professor Grace M. Bareis, Professor J. W. Bradshaw, Dr. R. W. Burgess, Professor W. D. Cairns, Professor Florian Cajori, Professor W. M. Carruth, Professor W. B. Carver, Mr. E. H.

Clarke, Dr. G. R. Clements, Professor F. N. Cole, Professor A. R. Crathorne, Professor C. H. Currier, Professor F. F. Decker, Professor L. L. Dines, Professor L. W. Dowling, Professor John Eiesland, Professor L. P. Eisenhart, Professor T. M. Focke, Professor Tomlinson Fort, Dr. M. G. Gaba, Professor D. C. Gillespie, Professor O. E. Glenn, Professor W. C. Graustein, Professor C. F. Gummer, Professor A. M. Harding, Professor H. E. Hawkes, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor F. H. Hodge, Professor E. V. Huntington, Professor W. A. Hurwitz, Dr. R. A. Johnson, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor H. G. Keppel, Professor G. A. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor F. R. Moulton, Professor A. D. Pitcher, Professor L. C. Plant, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor Maria M. Roberts, Professor E. D. Roe, Jr., Professor D. A. Rothrock, Professor J. E. Rowe, Professor C. H. Sisam, Professor P. F. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor E. B. Stouffer, Professor M. O. Tripp, Professor D. T. Wilson, Professor B. F. Yanney, Professor J. W. Young.

At the opening session on Tuesday morning Professor Focke occupied the chair, which was filled in succession by Professors Hedrick, Cajori, G. A. Miller, and Eisenhart. Professor Hedrick presided at the joint session. The Council announced the election of the following persons to membership in the Society: Dr. W. L. Crum, Yale University; Professor T. J. Fitzpatrick, University of Nebraska; Dr. T. R. Hollcroft, Columbia University; E. L. Ince, M. A., Trinity College, Cambridge, England; Mr. L. S. Odell, Manual Training High School, Brooklyn, N. Y.; Dr. T. A. Pierce, Harvard University. Five applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Professor Arnold Emch: "On the invariant net of cubics in the Steinerian transformation."

(2) Professor J. E. Rowe: "Theorems related to a point projection of the rational plane cubic curve."

(3) Professor J. E. Rowe: "Closed hexagons related to the rational plane cubic curve."

(4) Professor J. E. Rowe: "The projections of certain points upon the rational plane quartic curve."

(5) Professor TOMLINSON FORT: "Some theorems of comparison and oscillation."

(6) Professor O. D. KELLOGG: "Oscillation and interpolation properties of solutions of integral equations."

(7) Professor A. B. COBLE: "Finite groups determined by $2p + 2$ points in S_p."

(8) Dr. M. G GABA: "Complete existential theory of the postulates of the linear order η."

(9) Professor L. L. DINES: "The bordered Fredholm determinant and the related group of functional transformations."

(10) Professor R. G. D. RICHARDSON: "Contributions to the study of oscillation properties of ordinary linear differential equations of the second order."

(11) Professor C. N. MOORE: "On the summability of the developments in Bessel's functions."

(12) Professor G. A. MILLER: "Groups formed by special matrices."

(13) Professors VIRGIL SNYDER and F. R. SHARPE: "On the space involution of order 8 defined by a web of quadric surfaces."

(14) Dr. R. W. BURGESS: "A second approximation for cantilevers."

(15) Professor FLORIAN CAJORI: "L. Wantzel."

(16) Dr. G. M. GREEN: "Conjugate nets with equal point invariants."

(17) Dr. G. M. GREEN: "Plane nets with equal invariants."

(18) Professor FLORIAN CAJORI: "Newton's solution of numerical equations by the use of slide rules."

(19) Professor L. P. EISENHART: "Transformations of planar nets with equal invariants."

(20) Dr. L. C. MATHEWSON: "On the group of isomorphisms of a certain extension of an abelian group."

(21) Professor E. D. ROE, JR.: "Some restricted developments."

(22) Professor E. D. ROE, JR.: "A geometric representation. Second paper."

(23) Professor E. D. ROE, JR.: "Integral functions as products."

(24) Mrs. E. D. ROE, JR.: "Interfunctional expressibility problems of symmetric functions. Second paper."

(25) Professor E. L. DODD: "The approximation or graduation of a mortality table by means of a sum of exponential functions."

(26) Professor D. C. GILLESPIE: "Repeated integrals."

(27) Professor W. A. HURWITZ: "An expansion theorem for systems of linear differential equations."

(28) Professor W. C. GRAUSTEIN: "Note on isogenous complex functions of curves."

(29) Dr. MARY F. CURTIS: "A proof of the existence of the functions of the elliptic cylinder."

(30) Professor JOHN EIESLAND: "A Plücker geometry of flats in odd n-space."

(31) Professor H. J. ETTLINGER: "Theorems of oscillation for a generalized Sturmian boundary problem."

(32) Professor H. J. ETTLINGER: "Theorems of oscillation for the general real, self-adjoint system of the second order."

(33) Professor E. V. HUNTINGTON: "Bibliographical note on the use of the word mass in current textbooks."

(34) Professor L. P. EISENHART: "Darboux's contribution to geometry."

Dr. Curtis was introduced by Professor Bôcher. Dr. Green's second paper was presented by Professor Eisenhart, and Mrs. Roe's paper by Professor Roe. The papers of Professor Emch, Professor Coble, Dr. Mathewson, Professor Dodd, Professor Eiesland, and Professor Ettlinger, and Dr. Green's first paper were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond with the titles in the list above.

1. Considering $A_1(1, 0, 0)$; $A_2(0, 1, 0)$; $A_3(0, 0, 1)$; $E(1, 1, 1)$ as the fundamental triangle and unit point of a system of projective coordinates, the formulas for the Steinerian transformation associated with the quadrangle $A_1A_2A_3E$ are

$$\rho x_1' = x_1(x_2 + x_3 - x_1),$$
$$\rho x_2' = x_2(x_3 + x_1 - x_2),$$
$$\rho x_3' = x_3(x_1 + x_2 - x_3).$$

The net of cubics invariant under this transformation is represented by

$$a_1x_2x_3(x_2 - x_3) + a_2x_3x_1(x_3 - x_1) + a_3x_1x_2(x_1 - x_2) = 0.$$

Every triad of values $a_1 : a_2 : a_3$ defines a cubic of the net through the quadrangle and its diagonal points. Every cubic has the interesting property that the tangents at A_1, A_2, A_3, E meet in a point of the cubic with the coordinates a_1, a_2, a_3. Professor Emch discusses some geometric properties of this net of cubics.

2. In Professor Rowe's first paper four theorems are established, the first two of which are summarized in the statement: If $\varphi_{t_1}{}^2\varphi_{t_2}{}^2 = 0$ is the equation of the line determined by the two points t_1, t_2 of the rational plane cubic curve R^3, the discriminant of this quadratic in t_2, equated to zero, yields a quartic in t_1 whose roots are the parameters of the residual points cut out of the R^3 by the four tangents to the R^3 from the point x_i $(i = 0, 1, 2)$ of the plane; or for a fixed value of t_1 this same equation is the equation of a tritangent conic of the R^3 whose points of contact are t_1 and the points of contact of the two tangents that may be drawn from t_1 to the R^3. Theorems III and IV follow.

III. The parameters of the points of contact of the two tangents to the R^3 from any point of itself are harmonic to the parameters of the node.

IV. If t_2, t_3 are the parameters of the points of contact of the two tangents to the R^3 from the point t_1, and t_1' is the third point of the R^3 collinear with t_2, t_3, the transformation which sends t_1 into t_1' is involutory.

3. In this paper Professor Rowe derives one form of the necessary and sufficient condition that six points of the rational plane curve R^3 of order three lie on a conic, and uses this condition to prove the following theorems:

I. If six points of the R^3 lie on a conic, the tangents to the R^3 at these six points cut out of the R^3 six points on a conic.

II. If six points of the R^3 lie on a conic, the sides of the closed hexagon formed with these points as vertices cut the R^3 in six points on a conic.

III. If from a point of the R^3 the two tangents are drawn to the R^3 and from each point of contact of these two tangents the other two tangents to the R^3 are drawn, the six points of contact of the six tangents lie on a conic.

By a limit process it is evident that Theorem I may be derived as a corollary of Theorem II, although a different proof is given for each.

Also, it may be verified that the six points of contact of the three pairs of tangents drawn to the R^3 from three of its collinear points do not, in general, lie on a conic.

4. In this paper Professor Rowe proves the following four theorems together with three corollaries:

I. The parameters of the points of contact of the two tangents that can be drawn to the rational plane quartic curve R^4 from one of its nodes are harmonic to the parameters of the residual points cut out of the R^4 by any line through this node.

II, III, IV are summarized in the statement: If t_1, t_2 are the parameters of the points of contact of two flex tangents, of two double tangents, or of a flex tangent and a double tangent of the R^4, which cut out of the R^4 the residual points t_1', t_2', and if t_1'', t_2'' are the parameters of the points of contact of the other two tangents that can be drawn to the R^4 from the intersection of the tangents at t_1, t_2, the pairs (t_1, t_2), (t_1', t_2'), (t_1'', t_2'') form a set in involution.

5. Let y denote a solution of the differential equation $d(k(x)y')/dx + G(x)y = 0$. Professor Fort develops a number of theorems relative to $\alpha_1 k(x)y' + \alpha_2 y \equiv V(x)$ which reduce to well known theorems of comparison, oscillation, etc., in case that $V(x) \equiv y$.

6. In the *American Journal of Mathematics* for January, 1916, Professor Kellogg showed that continuity and orthogonality of a function set on the interval (0, 1) was insufficient to insure oscillation, but that if to these assumptions is added the hypothesis (D): the determinants $|\varphi_i(x_j)|$ are positive for $0 < x_0 < x_1 < x_2 \cdots < 1$, a considerable number of the properties of the more common sets of orthogonal functions follow. The present paper derives a sufficient condition on the kernel of an integral equation in order that its characteristic solutions may have the property (D) and its consequences.

7. It is the purpose of Professor Coble's paper (to appear in the *American Journal of Mathematics*) to show that a set

of $2p+2$ points in a projective space S_p, which is such that the quadrics on $2p+1$ of the points will pass through the remaining one, will define by purely projective processes a finite group which is isomorphic with the group determined by the odd and even theta functions in p variables. Since both the number of absolute constants of the point set and the number of moduli of the general theta function is $\frac{1}{2}p(p+1)$, this indicates an algebraic connection between such point sets and the general theta modular functions. This connection has been set forth hitherto only in the particular cases, $p=1, 2, 3$, and in the hyperelliptic case for all values of p; although in an earlier paper of the writer a grouping of the theta characteristics isomorphic with a grouping of the points of the set has been exhibited. It appears thereby that the point set may serve as the basis for a geometrical classification of the general theta functions, just as the algebraic curve of genus p has served for the discussion of the Riemannian theta functions of genus p.

8. In the March, 1917, BULLETIN, Professor E. V. Huntington gave three sets of completely independent postulates for serial order. His set A involved four postulates, which is as high a number as has been proved to be completely independent. In the present paper Dr. Gaba states seven postulates for the linear order η which form a categorical and completely independent set. For his 128 independence examples he gives 16 sets of points and 8 definitions of $<$ such that each definition is applicable to every one of the 16 sets of points and each combination of set and definition of $<$ yields a different example.

9. The determinant of the projective transformation

$$\phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy}{\delta + \int_0^1 \epsilon(y)\phi(y)dy},$$

is defined by Professor Dines to be the bordered Fredholm determinant

$$B = D\left[\delta - \int_0^1 \epsilon(y)\bar{\alpha}(y)dy\right] + \int_0^1\int_0^1 \epsilon(x)D_1(x, y)\bar{\alpha}(y)dxdy,$$

where $\bar{\alpha}(x) = \alpha(x)/\beta(x)$, and D and $D_1(x, y)$ are respectively the Fredholm determinant and first minor of the kernel $\gamma(x, y)/\beta(x)$. The following results are obtained:

(1) The determinant of the product of two projective transformations is equal to the product of their determinants.

(2) If the determinant B of a projective transformation is different from zero, a unique inverse transformation exists and is expressible in terms of B and four suitably defined first minors of B.

(3) The group of non-singular projective transformations (those for which $B \neq 0$) contains every finite transformation generated by a regular infinitesimal projective transformation as defined by Kowalewski; and, conversely, any non-singular projective transformation which does not differ too greatly from the identity transformation can be generated by such an infinitesimal transformation.

10. The Sturmian problem of determining the existence of parameter values λ for which the equation

$$(py_x)_x + G(x, \lambda)y = 0, \qquad p(x) > 0,$$

has solutions $y(x)$ satisfying linear self-adjoint boundary conditions has been much studied on account of its importance in mathematical physics and because it is a typical linear problem. Professor Richardson announces the results of further investigations as follows: (1) the derivation of sufficient conditions to be imposed on the function $G(x, \lambda)$ to insure the existence of solutions oscillating as many times as desired; (2) the determination of conditions on $G(x, \lambda)$ such that for a given interval of λ there is precisely one or precisely two parameter values for which exist solutions satisfying the boundary conditions and oscillating a prescribed number of times; (3) the completion of the theory of oscillation for the most important special case $G(x, \lambda) \equiv q(x) + \lambda k(x)$, viz., an investigation of the only case not hitherto studied in detail. For the non-definite case ($k(x)$ both signs, $q(x)$ positive in at least a part of the x-interval) it is found that there exist two integers ν_1, ν_2 ($\nu_2 \geqq \nu_1 \geqq 0$) such that if n denote the number of oscillations of the solution then for $n < \nu_1$ there are no solutions, for $n > \nu_1$ there are at least two, and for $n \geqq \nu_2$ there are precisely two.

11. The principal theorems obtained in Professor Moore's paper are the following: (A) If $f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, approaches a finite limit as x approaches $+ 0$, and furthermore is such that $(f(x) - f(+ 0))/x^\delta$, where $\delta > \frac{1}{2}$, remains finite in the neighborhood of the origin, then the development of $f(x)$ in Bessel's functions of order $\nu (\nu \geqq 0)$ will be summable $(C, k > \frac{1}{2})$ to the value $f(+ 0)$ for $x = 0$, provided $\nu = 0$ or $f(+ 0) = 0$, and will be uniformly summable $(C, k > \frac{1}{2})$ to the value $f(x)$ in any interval $0 \leqq x \leqq c \leqq 1$, throughout which $f(x)$ is continuous, provided $\nu = 0$ or $f(+ 0) = 0$. (B) If $\sqrt{x} f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, then the development of $f(x)$ in terms of Bessel's functions of order $\nu (\nu \geqq 0)$ will be summable $(C, k > 0)$ to the value $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at every point in the interval $0 < x \leqq 1$ at which $f(x)$ is continuous or has a finite jump, and will be uniformly summable $(C, k > 0)$ to $f(x)$ throughout any closed interval lying entirely in the interval $0 < x \leqq 1$ and throughout which $f(x)$ is continuous.

12. Any set of square matrices of order n having one and only one unity element in each row and in each column, while all their other elements are zeros, generates a finite group which is simply isomorphic with some substitution group of degree n. When these unity elements are replaced by kth roots of unity the resulting matrices generate a group which is simply isomorphic with a substitution group of degree kn. When this substitution group is transitive it has n systems of imprimitivity, each system being of degree k. The main object of Professor Miller's paper is to consider the nature of these elements in order that any imprimitive group of degree kn having n systems of imprimitivity can be represented by such square matrices of order n. The following theorems are established: Every possible imprimitive group of degree $2n$ which has n systems of imprimitivity is simply isomorphic with a group formed by square matrices of order n having one and only one ± 1 element in each row and in each column while all their other elements are zeros. If an imprimitive group of degree kn has for its head H the direct product of n regular cyclic groups of order k, then every invariant imprimitive subgroup has for its head a subgroup of H which is simply isomorphic with the direct product of $n - 1$ cyclic groups of order k and of some subgroup of such a cyclic group.

13. The involution discussed by Professors Snyder and Sharpe is defined by associating the planes of one space with the quadric surfaces of a web in another. We have a (1, 8) correspondence between the points of the two spaces, and an involution of order 8 in the space of the quadrics. The surface of coincident points, the surface of branch points, and the residual surface are obtained and described. The dual of the surface of branch points is the symmetroid, the characteristic properties of which are easily obtained by this method. As particular cases we meet with the Weddle surface, the Kummer surface, the Hessian of the cubic, and all of the line congruences of order 2.

14. The engineer uses constantly certain simple formulas for the deflection of a horizontal beam fixed at one end and bent by a vertical force at the other, and for the angle of slope of the free end. The treatment by which these formulas are derived can not readily be modified to give true results for large deflections; but by using the equation for the real elastic curve and developing in series the elliptic integrals involved, Dr. Burgess obtains the following formulas for the deflection d, the horizontal component t of the displacement of the free end, and the angle of slope ψ:

$$d = \frac{l^3W}{3EI} - \frac{4}{105}\frac{l^7W^3}{E^3I^3} + .008\frac{l^{11}W^5}{E^5I^5}$$

$$t = \frac{1}{15}\frac{l^5W^2}{E^2I^2} - \frac{4}{315}\frac{l^9W^4}{E^4I^4}$$

$$\tan\psi = \frac{1}{2}\frac{l^2W}{EI} - \frac{1}{240}\frac{l^6W^3}{E^3I^3} + \frac{l^{10}W^5}{600E^5I^5},$$

where E is the constant of elasticity, W the deflecting force, l the length of the beam, and I the geometric moment of inertia of a cross-section about an axis perpendicular to the plane of bending. The first terms in the expressions for d and $\tan\psi$ are the engineers' formulas; the other terms give corrections important for larger deflections, and apparently sufficient within the elastic limit.

15. Professor Cajori gave some details on the life and work of Pierre-Laurent Wantzel (1814–1848) of the Polytechnic School in Paris.

16. In his first paper, Dr. Green gives a new geometric characterization of conjugate nets with equal point invariants, which is a refinement of a theorem established by him in a previous paper.* The following theorem is first proved geometrically. Let t_1 and t_2 be the ray tangents at a point of a surface S referred to a conjugate net. On the corresponding ray there are two focal points F_1 and F_2, corresponding respectively to the ray curves of which t_1 and t_2 are the tangents. Then the tangent to S which is conjugate to t_1 meets the ray in F_2, and the tangent conjugate to t_2 meets the ray in F_1.

It is then proved that a necessary and sufficient condition that a conjugate net have equal point invariants is that the ray tangents meet the ray in the focal points of the ray, each ray tangent passing through the focal point which does not correspond to it.

If the ray tangents meet the ray in the corresponding focal points of the ray, then the ray curves coincide with the asymptotic net, and the conjugate net has not equal invariants.

17. The geometric characterization of conjugate nets with equal point invariants given in Dr. Green's first paper does not admit of immediate extension to the case of planar nets with equal invariants. For any planar net N, the ray of a point may be defined as the line joining the minus first and first Laplace transforms of the point, but every one-parameter family of rays forms a developable and gives rise to a focal point on each ray. The first object of Dr. Green's second paper is to define a net N' congruentially associated with the net N. Let t_1, t_2 be the tangents to the curves of N at a point of the plane, and let t_1', t_2' be the tangents to the curves of N' at the same point. The curve of N' to which t_1' is tangent gives rise to a focal point F_1 on the ray of the net N. Similarly, a focal point F_2 on this ray corresponds to the tangent t_2'. The net N' is said to be congruentially associated with N if the harmonic conjugate of t_1' with respect to t_1, t_2 passes through F_2, and the harmonic conjugate of t_2' with respect to t_1, t_2 passes through F_1. It is proved that there always exists one and only one net congruentially associated with the given one.

A necessary and sufficient condition that the net N have equal invariants is that at every point the tangents to the two curves of N separate harmonically the tangents to the two curves of the congruentially associated net N', or, in other words, that each ray of N be met by the tangents of N' in its focal points with respect to N'.

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* *Amer. Jour. of Mathematics*, vol. 38 (1916), p. 313.

18. Professor Cajori notes three accounts of Newton's solution of numerical equations by the use of logarithmic slide rules, only one of which is given in the collected works of Newton and is referred to in the Encyclopédie des Sciences mathématiques, Tome I, Volume 4, page 429, namely, four sentences in a letter of Oldenburg to Leibniz of June 24, 1675. A second fuller account is described in E. Stone's New Mathematical Dictionary, second edition, London, 1743 and is reprinted in Cajori's History of the Slide Rule, 1909. A third account, purporting to be taken directly from Newtonian manuscripts, is printed in James Wilson's Mathematical Tracts of the late Benjamin Robins, London, 1761. The three accounts contain four slightly different modes of procedure.

19. In the January number of the *Transactions* Professor Eisenhart developed a theory of transformations of a conjugate system of curves on any surface into conjugate systems on other surfaces. These results can be extended so as to give transformations of conjugate nets on two-dimensional spreads in space of any order. In the present paper he applies them to nets in the plane, and in particular to nets with equal point invariants. In the June number of the *Annals of Mathematics* it was shown how by quadratures one can obtain from such a net a system of asymptotic curves on a surface which are perspective with the net from a point. Two nets with equal invariants in the relation of a transformation here discussed are found to be perspective with two surfaces which are the focal surfaces of a W-congruence. The planar nets of period three under transformations of Laplace have equal invariants. The paper deals also with transformations of these nets into nets of the same kind, and establishes the existence of a theorem of permutability of these transformations. The analysis involved in this investigation is capable also of interpretation as giving transformations of quadrics, certain ruled surfaces, and surfaces of the kind first discussed by Tzitzeica into surfaces of these respective types.

20. In 1908 Professor G. A. Miller showed that "if an abelian group H which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of H into its inverse, then the group of isomorphisms of this extended group is the holomorph of H."[*] The present paper by Dr. Mathewson discusses an elaboration of Professor Miller's theorem. In toto it is proved that if G is formed by extending an abelian group H which has operators of order > 2 by a certain operator which transforms every one of its operators into the same power of itself and which is commutative with no operator of odd order in H, then the group of isomorphisms of G is the holomorph of H, and is a complete group if H is of odd order.[†] In the proofs of the various theorems of the paper free use is made of properties of rational integers, which are either quoted or established.

21. In this paper in an elementary way without the assumption of Fourier's series, Professor Roe derives the two developments

$$f(x) = f(a) + \sum_{r=1}^{r=\infty} \int_a^x (-1)^{r-1} \frac{\sin(2rf'(x))}{r}\, dx,$$

where the range of a and x is determined by

$$\pi/2 \geqq f'(x) \geqq -\pi/2;$$

and

$$f(x) = f(a) + \frac{\pi}{2}(x-a) - \sum_{r=1}^{r=\infty} \int_a^x \frac{\sin(2rf'(x))}{r}\, dx,$$

where the range of a and x is determined by

$$\pi \geqq f'(x) \geqq 0.$$

In both cases the range is varied by suitable transformations. Numerous applications to the development of functions and the summations of certain series are given.

22. This paper is a continuation and further elaboration of matters suggested in Professor Roe's previous paper, presented at the October, 1916, meeting.

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[*] Miller, *Philosophical Magazine*, vol. 231 (1908), p. 224.

[†] Miller, *Mathematische Annalen*, vol. 66 (1908), p. 135.

The interpretation of the complex values of y in $y = f(x, z)$, surface, is given, and it is shown how the equations of surfaces may be expressed by the method considered, and how the equations of families of spirals in two coordinates on surfaces are derived. The spiral systems arising from the complex values of y in $y = \{x^{x-1}(x-1)^x\}^{1/(1-2x)}$ together with the value systems of this function are considered in detail. Special illustrations are given.

23. Professor Roe gives in this paper from an elementary standpoint the necessary and sufficient conditions for developing integral functions into products, and thus furnishes a method for developing into products series whose zeros are known or can be found, the results of which agree with those found by the methods of the theory of functions. Illustrations are furnished.

24. In this paper Mrs. Roe continues the investigation begun in her paper presented at the October, 1916, meeting. From the product and quotient tables she derives new formulas for the coefficients of each of the complexes involved, in terms of each of the others, together with new proofs of symmetry in certain of the tables of such coefficients.

25. For the Makeham approximation or graduation the English H^m Mortality Table was divided at the age 28. Professor Dodd finds it convenient to divide the American Experience Table at the age 70, and to approximate each portion separately by means of a more simple function, viz.,

$$a + br^x + cs^x.$$

This function gives very closely the number of survivors at the age x. The terms before and after multiplication by the powers of the discounting factor give rise to geometric progressions, making easy the computation of the life functions—including increasing life functions—at any desired rate of interest. This assists in the problem of determining the interest rate which the insured actually realizes, on the basis of the gross premium. In constructing or graduating a table by the above function from raw material, the least square method may be employed readily, after first approximations have been obtained.

26. The theorem of the note by Professor Gillespie is: the repeated integral of the bounded function $f(x, y)$ may be taken in either order with equal results; i. e.,

$$\int_0^1 dx \int_0^1 f(x, y)dy \quad \text{and} \quad \int_0^1 dy \int_0^1 f(x, y)dx$$

are equal where they both exist.

27. Professor Hurwitz presents an analogue to the Sturm-Liouville theory, leading to the expansion of two given functions in terms of solutions of a pair of linear ordinary differential equations with linear parameter, satisfying certain linear boundary conditions, the coefficients in the two expansions being the same.

28. In this note Professor Graustein proves the following theorem in connection with Volterra's theory of isogenous complex functions of curves: If the gradients c_1, c_2 of the functions F_1, F_2 in $F = F_1 + iF_2$, where $F_1 = F_1|[L]|$, $F_2 = F_2|[L]|$ are functions of the first degree of the space curve L, are in general analytic, the gradients γ_1, γ_2 of Φ_1, Φ_2 in $\Phi = \Phi_1 + i\Phi_2$, an arbitrary complex function of L of the first degree isogenous with F, are analytic save perhaps in points of singularity of c_1 or c_2 and points in which both these vectors are indeterminate.

29. Heine's proof of the existence of the functions of the elliptic cylinder, that is, of the existence of real periodic solutions of period 2π of the differential equation

$$\frac{d^2E(\varphi)}{d\varphi^2} + 4\left(\frac{2}{b}\cos 2\varphi + z\right)E(\varphi) = 0, \qquad b \neq 0,$$

—a differential equation fundamental in physical problems dealing with elliptic cylinders and elliptic membranes—is clearly at fault in an essential point. Dannacker's attempt to remedy the error is unsatisfactory. In the present paper Dr. Curtis sets up the proof in a rigorous fashion.

30. In Professor Eiesland's paper an attempt has been made to develop a geometry of flats in odd n-space on the same lines as Plücker's geometry in 3-space. Rectangular coordi-

nates are used throughout and some interesting results are obtained. The subject has of necessity led to certain projective properties of n-space which are believed new. The flat-complex seems to contain far-reaching possibilities when completely developed. One thinks especially of the interpretation of flat-geometry as sphere-geometry on the one hand and on the other as anallagmatic geometry in the next higher space, such as has been treated by Professor Ranum.

31. Professor Ettlinger generalizes the Bôcher-Sturm oscillation theorem to the following system:

$$d[K(x, \lambda)u_x(x, \lambda)]/dx - G(x, \lambda)u(x, \lambda) = 0,$$

$$L[a, u(a, \lambda)] = 0, \quad L[b, u(b \ \lambda)] = 0,$$

where

$$L[x, u(x, \lambda)] = \alpha(x, \lambda)u(x, \lambda) - \beta(x, \lambda)K(x, \lambda)u_x(x, \lambda).$$

In general the same conditions are placed on the coefficients of the differential equation and the boundary conditions* except that $\alpha(a, \lambda)/\beta(a, \lambda)$ and $-\alpha(b, \lambda)/\beta(b, \lambda)$ decrease as λ increases, and if $\beta(x, \lambda) \not\equiv 0$ and

$$\{\alpha\beta\} = \alpha_x(x, \lambda)\beta(x, \lambda) - \alpha(x, \lambda)\beta_x(x, \lambda) + \alpha^2(x, \lambda)/K - \beta^2(x, \lambda)G,$$

then $\lim (\lambda \rightarrow \Lambda_1) \max \{\alpha\beta\} < 0$ and $\{\alpha\beta\}$ changes sign once and only once as λ increases. The existence of an infinite set of characteristic numbers for this system is proved and oscillation theorems for $L[x, u_p(x)]$ and $u_p(x)$ are established. The Sturm-Bôcher theorem appears as a special case of this result, where a particular form is assigned to $\alpha(x, \lambda)$ and $\beta(x, \lambda)$.

32. In a forthcoming paper in the *Transactions* Professor Ettlinger extends the work of Birkhoff concerning the solutions of a general self-adjoint linear system of the second order. The object of the present paper is to generalize these results to a system with more general boundary conditions, and to derive more exact oscillation theorems.

Consider the second order linear differential equation

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* See Bôcher, Leçons sur les Méthodes de Sturm, Paris, 1917, p. 63.

$$d[K(x, \lambda)u_x(x, \lambda)]/dx - G(x, \lambda)u(x, \lambda) = 0,$$

$$L_i[a, u(a, \lambda)] = L_i[b, u(b, \lambda)] \qquad (i = 1, 2),$$

where

$$L_i[x, u(x, \lambda)] = \alpha_i(x, \lambda)u(x, \lambda) - \beta_i(x, \lambda)K(x, \lambda)u_x(x, \lambda).$$

Under the corresponding sets of conditions, which are the natural extensions of those imposed in the earlier paper, the existence of an infinite set of parameter values for this system is established and an exact oscillation theorem obtained for $L[x, u_p(x)]$ and a slightly less exact theorem for the oscillations of $u_p(x)$.

The case considered above yields the other (see the earlier paper) by assigning a special form to the coefficients of the boundary conditions.

The work is based on the generalized Sturmian theorem of oscillation (see the preceding abstract).

33. As a result of an examination of the definition of mass in a large number of current books on mechanics and physics, Professor Huntington finds the following principal ways in which the mass of a body is said to be measured: (1) by force per acceleration (inertia); (2) by the beam balance (standard weight); (3) by mutual acceleration (Mach's definition); (4) by the number of supposed identical particles in the body. The pseudo-definition of mass as "quantity of matter," and various circular definitions, are condemned. The conditions for measurability in general are stated, and suggestions in regard to the best use of the term mass are discussed. The paper will be offered to the *American Mathematical Monthly*.

34. Professor Eisenhart's address, delivered before a joint session of the Society and the Mathematical Association, was devoted to an analysis of Darboux's contributions to geometry. This address will be published in the January issue of the BULLETIN.

F. N. COLE,
Secretary.

IRRATIONAL TRANSFORMATIONS OF THE GENERAL ELLIPTIC ELEMENT.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 28, 1917.)

In 1865 G. G. A. Biermann published as derived from Weierstrass's lectures the following formula:

$$(1)\quad F(x) = x_0 + \frac{\sqrt{R(x_0)}\,\sqrt{S} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2\left[s - \frac{1}{24}R''(x_0)\right]^2 - \frac{1}{2}A\cdot R(x_0)}.$$

F is the solution of

$$(2)\quad (F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).$$

The accents used with F and R denote differentiation, x_0 is an arbitrary constant, and A, B, C, B', A' are constant coefficients. Also

$$(3)\quad S = 4s^3 - g_2 s - g_3 = 4(s - e_1)(s - e_2)(s - e_3),$$

$$s = \wp(x),\quad g_2 = AA' + 3C^2 - 4BB',$$

$$g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3.$$

In Enneper, Elliptische Functionen, Greenhill, Elliptic Functions; and Haentzschel, Reduction der Potentialgleichung, are several applications of this formula.

Replacing x by $t + iu$ and F by $x + iy$ in the preceding equations gives

$$(4)\qquad x + iy = \varphi(t + iu),$$

from which may be obtained in the usual manner two mutually orthogonal families of curves having t and u as parameters. For several cases this has been done by Haentzschel, and the writer has given discussions and extensions in *Archiv der Mathematik und Physik*. It is evident that (1) becomes very simple when x_0 is a root of $R(x) = 0$, for which Haentzschel's results may be consulted. When x_0 is infinite Enneper obtains from (1)

$$(5) \qquad v = \frac{\sqrt{A}\ \sqrt{S} + 2Bs - BC + AB'}{2(B^2 - AC) - 2As},$$

but does not develop the results. However, the rationalization of (5) gives

$$(6) \quad s^2 - s(Av^2 + 2Bv + C) + (B^2 - AC)v^2 + (BC - AB')v + (C^2 - AA')/4 = 0.$$

Next placing x equal to 0 in (1),

$$(7) \quad s^2v^2 - s(A' + 2B'v + Cv^2) + (B'^2 - A'C) + (B'C - A'B)v + (C^2 - AA')v^2/4 = 0,$$

which becomes (6) when v is replaced by $1/v$, noticing the interchange of constants required by (2). Later in this paper reference will be made to (7) in another discussion.

In using (1) to obtain orthogonal curves it is of great advantage to make the following changes: $F(x)$ is replaced by $x_0 + v$ and (2) is written

$$(8) \qquad (v')^2 = av^4 + 4bv^3 + 6cv^2 + 4dv + e \equiv R(x_0 + v)$$

$$[v = x' + iy'],$$

whence

$$(9) \qquad a = \frac{R''''(x_0)}{4!} = A, \quad 4b = \frac{R'''(x_0)}{3!}, \quad 6c = \frac{R''(x_0)}{2!},$$

$$4d = R'(x_0), \quad e = R(x_0),$$

$$g_2 = ae - 4bd + 3c^2, \quad g_3 = ace + 2bcd - ad^2 - eb^2 - c^3.$$

The rationalized form of (1) thus becomes

$$(10) \quad s^2v^2 - s(cv^2 + 2dv + e) + (d^2 - ce) + (cd - be)v + (c^2 - ae)v^2/4 = 0,$$

which is of the same form as (7).

Greenhill, on page 151 (l. c.), refers to (1) as a method of reducing the general elliptic element to the Weierstrass form. Now in using (1) as a solution of (2) it is required that s shall satisfy

$$(11) \qquad (s')^2 = 4s^3 - g_2s - g_3,$$

so that one value is of course $\wp(x)$. Since (11) is only a special case of (2) the corresponding form of (1) gives the solution

$$\bar{s} = \frac{\sqrt{(-g_3)}\ \sqrt{S} - \tfrac{1}{2}\, g_2 s - g_3}{2s^2} \tag{12}$$

or in rational form,

$$s^2\bar{s}^2 + g_2 s\bar{s}/2 + g_3(s + \bar{s}) + g_2^2/16 = 0. \tag{13}$$

By elimination of s between (10) and (13) there is obtained a new equation involving v and $\bar{s}$ which is a still more complicated solution of (8), or in other words a method of reducing the general elliptic element to the Weierstrass form. Because v and $\bar{s}$ do not enter symmetrically this equation is too complicated to be of interest.

The conjugate of (10) formed by changing s and v into σ and w respectively provides a transformation of the general into the Weierstrass form and vice versa, so that making s and σ identical, and then eliminating s gives an identical transformation of the general elliptic element. After cancellation of the trivial factor $(v - w)^2$ the result is

$$\begin{aligned} v^2w^2(b^2e^2 - aed^2) + vw(v + w)(3bce^2 - 2bed^2 - ade^2)& \\ + vw\left(\frac{9c^2e^2}{2} - 2bde^2 - 2d^4 - \frac{ae^3}{2}\right)& \\ + (v + w)(3cde^2 - be^3 - 2d^3e)& \\ + (v^2 + w^2)\left(\frac{9c^2e^2}{4} - 3cd^2e + d^4 - \frac{ae^3}{4}\right) &= 0. \end{aligned} \tag{14}$$

If s and σ in (10) and its conjugate are written $\wp(t + iu)$ and $\wp(t - iu)$ respectively, and u is eliminated, the resulting equation of the sixteenth degree is resolvable into one factor of the eighth degree and two equal factors each of the fourth degree having $\wp(2t)$ as a parameter. Now if $\wp(2t)$ is taken as $(d^2 - ce)/e$ the equal factors are identical with (14), which also, as is evident from the preceding, corresponds with $s - \sigma = 0$. Lastly the discriminant of (14) as a quadratic equation in v is

$$\begin{aligned} e(aw^4 + 4bw^3 + 6cw^2 + 4dw + e)(d^2 - ce - e\epsilon_1)& \\ \times (d^2 - ce - e\epsilon_2)(d^2 - ce - e\epsilon_3)&. \end{aligned} \tag{15}$$

UNIVERSITY OF PENNSYLVANIA,
April, 1917.

NOTE ON THE PARAMETRIC REPRESENTATION OF AN ARBITRARY CONTINUOUS CURVE.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society, April 28, 1917.)

A CONTINUOUS curve may be defined by a pair of equations

$$x = f(t), \qquad y = \varphi(t),$$

where the functions f and φ are defined and continuous throughout some interval, say for $0 \leqq t \leqq 1$, and are not both constant throughout that interval. It is sometimes an aid to the imagination to suppose that they are not simultaneously constant throughout any subinterval of their interval of definition. The question naturally arises, whether this implies a restriction on the curve itself, or merely on the particular parametric representation employed. It is obvious that if there are only a finite number of intervals where f and φ are constant together, a suitable change of parameter will eliminate such intervals entirely. The present note offers a proof that this is true in all cases. A decidedly different proof has been given by Fréchet;* it is believed that the method of treatment here adopted is considerably more elementary than his, as far as obtaining the indicated result is concerned, though it does not yield certain additional knowledge which is incidental to his presentation.

On careful scrutiny, it appears that the problem of a satisfactory definition of the identity of two curves is not an altogether simple one;† to avoid a discussion of this point, the result is stated in the following form, which is free of ambiguity:

THEOREM. *Let a pair of functions*

$$x = f(t), \qquad y = \varphi(t) \tag{1}$$

be defined and continuous for $0 \leqq t \leqq 1$. *Then it is possible to*

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* Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), pp. 1–74; pp. 58–59, 67–70.

† Cf. Fréchet, loc. cit., pp. 51–53.

define a function $\sigma(t)$, *single-valued, continuous, and non-decreasing for* $0 \leqq t \leqq 1$, *and a pair of functions* $F(\tau)$, $\Phi(\tau)$, *single-valued and continuous for* $\sigma(0) \leqq \tau \leqq \sigma(1)$, *so that the two latter functions are not simultaneously constant throughout any interval* $\tau_1 \leqq \tau \leqq \tau_2$, *where* $\tau_1 < \tau_2$, *and so that*

$$F(\sigma) = f(t), \qquad \Phi(\sigma) = \varphi(t),$$

identically for $0 \leqq t \leqq 1$.

The function σ will be so chosen that $\sigma(0) = 0$, whereby the interval of values for σ becomes $(0, \lambda)$, if $\lambda = \sigma(1)$; but this remark is unimportant, to the extent that the possibility of making such a choice of σ is clearly implied in the theorem as already stated, since the subtraction of a constant from $\sigma(t)$ does not affect any property of it with which the theorem is concerned; and it would be possible further to replace the value λ by 1.

As a consequence of the theorem, it is to be observed that within the limits $0 \leqq t \leqq 1$, $0 \leqq \sigma \leqq \lambda$, to every value of t corresponds one (and just one) value of σ, and to every value of σ corresponds at least one value of t, so that every point (x, y) given by the relations (1) is given by the equations

$$x = F(\sigma), \qquad y = \Phi(\sigma),$$

and vice versa. But it is not true in general that t is a single-valued function of σ.

To prove the theorem, let us notice first that it is practically obvious if the curve (1) is rectifiable, the arc-length s, defined in the usual way as limit of inscribed polygons, serving the purpose of σ. The function $s(t)$ is certainly single-valued, continuous, and non-decreasing.* To every value of s between 0 and l, where l is the total length of the curve, corresponds at least one value of t, in consequence of the properties just named, and so at least one point (x, y) of the curve. Furthermore, although t need not be a single-valued function of s, a specified value of s can not give more than one point (x, y), since the values of s for any two distinct points differ by at least the length of the chord joining the points. So x and y are single-valued functions of s, which may be denoted by $F(s)$ and $\Phi(s)$, defined for $0 \leqq s \leqq l$, and fulfilling the identities $F(s) = f(t)$, $\Phi(s) = \varphi(t)$, from the manner of their definition. They are continuous, since $\Delta x \leqq \Delta s$, $\Delta y \leqq \Delta s$; and they are evidently not constant together over any interval. A proof of the last statement, labored as the proof of an evident fact is likely to be, is as follows: Because of the monotone character of s, an interval of values of s corresponds to an interval of values of t. If F and Φ were constant throughout an s-interval, f and φ would be constant throughout the corresponding t-interval; but then this t-interval would give just a single point (x, y), and so a single value, not an interval of values, of s. So the theorem is established for any rectifiable curve.

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* Cf., e. g., Pierpont, Theory of Functions of Real Variables, vol. 2, p. 585.

Now consider an arbitrary continuous curve, given in the form (1). Let $\omega(\delta)$ be the maximum of

$$\sqrt{[f(t'') - f(t')]^2 + [\varphi(t'') - \varphi(t')]^2}$$

for $0 \leqq t' \leqq 1$, $0 \leqq t'' \leqq 1$, $|t' - t''| \leqq \delta$. This function is continuous and non-decreasing for all values of $\delta \geqq 0$. The latter property is evident; the former follows from the facts that

$$\lim_{\delta=0} \omega(\delta) = 0, \qquad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2).$$

The function $\omega(\delta) + \delta$ is continuous and increasing. Let ψ be the inverse function, so that

$$\delta = \psi[\omega(\delta) + \delta];$$

then ψ also is single-valued, continuous, and increasing, and $\psi(0) = 0$.

Let t be any point in the interval $(0, 1)$. Let $t_1, t_2, \cdots, t_{n-1}$ be any $n - 1$ points between 0 and t, arranged in order, and let $t_0 = 0$, $t_n = t$. Let

$$\delta_i = \sqrt{[f(t_i) - f(t_{i-1})]^2 + [\varphi(t_i) - \varphi(t_{i-1})]^2} \quad (i = 1, 2, \cdots, n),$$

and, finally, let

$$(2) \qquad \sigma(t) = \text{upper limit of } \sum_{i=1}^{n} \psi(\delta_i)$$

for all possible choices of $t_1, \cdots, t_{n-1}$ and all values of n. This function $\sigma(t)$ will be shown to fulfil the requirements of the theorem.

In the first place, $\sigma(t)$ is defined for $0 \leqq t \leqq 1$. For

$$\delta_i \leqq \omega(t_i - t_{i-1}) < \omega(t_i - t_{i-1}) + (t_i - t_{i-1}),$$

and consequently

$$\psi(\delta_\iota) < t_\iota - t_{\iota-1}$$

and

$$\sum_{\iota=1}^{n} \psi(\delta_\iota) < t,$$

so that the upper limit in (2) is finite. It is evident that $\sigma(t)$ can not diminish when t increases.

Let $\sigma(t', t'')$ denote the upper limit of the sums $\Sigma\psi(\delta_\iota)$ obtained by subdividing the interval (t', t''); in particular, $\sigma(0, t) = \sigma(t)$. The functions σ with one and two arguments respectively will not be in any danger of confusion. The proof that $\sigma(t)$ is continuous resembles the corresponding proof for rectifiable curves,* but it is necessary to exercise some care, for two reasons. In the first place, a sum $\Sigma\psi(\delta_\iota)$ may be *diminished* by introducing new points of subdivision and retaining the old ones; it may even happen that the greatest possible value of the sum is attained by taking $n = 1$, and in no other way.† Secondly, if $t' < t'' < t'''$, while it is evident that

$$\sigma(t', t''') \geqq \sigma(t', t'') + \sigma(t'', t'''),$$

the sign of equality need not hold.‡

Let c be any point in the interval $0 < c \leqq 1$. It will be shown first that $\sigma(t)$ is continuous on the left at the point c. Since σ is monotone increasing, it is required only to show that if ϵ is any positive quantity, there is a $t < c$ such that

$$\sigma(t) > \sigma(c) - \epsilon. \tag{3}$$

Take any subdivision of $(0, c)$ such that the corresponding sum $\Sigma\psi(\delta_\iota)$ is greater than $\sigma(c) - \frac{1}{2}\epsilon$. For $t > t_{n-1}$, let

$$\delta_n' = \sqrt{[f(t) - f(t_{n-1})]^2 + [\varphi(t) - \varphi(t_{n-1})]^2}.$$

If t is sufficiently near to c, δ_n' will differ so little from δ_n that§ $\psi(\delta_n') > \psi(\delta_n) - \frac{1}{2}\epsilon$, and the points $t_1, \cdots, t_{n-1}$ give a subdivision of $(0, t)$ such that the corresponding sum $\Sigma\psi(\delta_\iota)$ exceeds $\sigma(c) - \epsilon$. Hence (3) is satisfied.

In the same way, it may be shown that, for $0 \leqq c < 1$,

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* Cf., e. g., Lebesgue, Leçons sur l'Intégration, pp. 53–54, 60.

† E. g., let $f(t) = \sqrt{t} - t$, $\varphi(t) = 0$. Then $\omega(\delta) = \sqrt{\delta} - \delta$, $\psi(\delta) = \delta^2$, and $\sigma(1) = 1 = \psi(1)$, while $\Sigma\psi(\delta_\iota) < 1$ if $n > 1$.

‡ In the example of the preceding footnote, $\sigma(0, \frac{1}{2}) + \sigma(\frac{1}{2}, 1) = \frac{1}{2}$.

§ Of course it is not inconceivable that $\psi(\delta_n') > \psi(\delta_n)$.

$$\sigma(t, 1) > \sigma(c, 1) - \epsilon,$$

if t is greater than c and sufficiently near to c. Since

$$\sigma(c, 1) \geqq \sigma(c, t) + \sigma(t, 1),$$

it follows that

$$\sigma(c, t) < \epsilon;$$

but even now it can not be inferred immediately (except for $c = 0$) that $\sigma(t)$ is continuous on the right at c, because of the failure of the additive property for consecutive intervals.

Let ϵ be any positive quantity. Let $t > c$ be so close to c that $\sigma(c, t) < \frac{1}{3}\epsilon$, which is possible, in consequence of the preceding paragraph; and so close also that the length of the chord joining the points $[f(c), \varphi(c)]$ and $[f(t'), \varphi(t')]$ is less than $\frac{1}{3}\epsilon$ for $c \leqq t' \leqq t$. If δ' and δ'' are the lengths of the chords joining these two points with any third point, then $|\delta'' - \delta'| < \frac{1}{3}\epsilon$; and it follows that

$$|\psi(\delta'') - \psi(\delta')| < \tfrac{1}{3}\epsilon, \tag{4}$$

since any increment of $\omega(\delta) + \delta$ is greater than or equal to the corresponding increment of δ, because of the monotone character of ω, and any increment of ψ consequently is less than or equal to the corresponding increment of its argument.

After choice of a particular value of t satisfying the conditions just specified, take any subdivision of the interval $(0, t)$ such that the corresponding sum $\Sigma\psi(\delta_\iota)$ exceeds $\sigma(t) - \frac{1}{3}\epsilon$, the points of division being $t_0 = 0, t_1, \cdots, t_{n-1}, t_n = t$. Suppose k is the index such that $t_{k-1} < c \leqq t_k$. Then the terms $\psi(\delta_{k+1}) + \cdots + \psi(\delta_n)$ constitute a sum which can not exceed $\sigma(c, t)$, and so is less than $\frac{1}{3}\epsilon$. If δ_k' is the length of the chord joining $[f(t_{k-1}), \varphi(t_{k-1})]$ and $[f(c), \varphi(c)]$, then

$$\psi(\delta_k) - \psi(\delta_k') < \tfrac{1}{3}\epsilon,$$

in consequence of (4). Hence

$$\psi(\delta_1) + \cdots + \psi(\delta_{k-1}) + \psi(\delta_k') > \psi(\delta_1) + \cdots$$
$$+ \psi(\delta_n) - \tfrac{2}{3}\epsilon > \sigma(t) - \epsilon,$$

and it follows that

$$\sigma(c) > \sigma(t) - \epsilon,$$

which proves the continuity of $\sigma(t)$ on the right at c.

The rest of the proof of the theorem is almost immediate. In consequence of the continuity of $\sigma(t)$ there corresponds to every value of σ between 0 and $\lambda = \sigma(1)$ at least one value of t, and so at least one point of the curve. There can not be more than one point for a single value of σ, because the values of σ corresponding to any two distinct points differ by at least $\psi(\delta)$, if δ is the length of the chord joining the points. If we set $x = F(\sigma)$, $y = \Phi(\sigma)$, these are single-valued functions of σ, and are identically equal to $f(t)$ and $\varphi(t)$ respectively, by their very definition. They are continuous, because

$$\Delta\sigma \geqq \psi(\sqrt{\Delta x^2 + \Delta y^2}), \qquad \Delta x \leqq \sqrt{\Delta x^2 + \Delta y^2} \leqq \omega(\Delta\sigma) + \Delta\sigma,$$

and similarly for Δy. Finally they are not constant together in any interval, by the same argument as in the case of a rectifiable curve.

The extension of the work to curves in three or more dimensions requires only the writing down of a correspondingly larger number of symbols.

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JOHN WALLIS AS A CRYPTOGRAPHER.

BY PROFESSOR DAVID EUGENE SMITH.

It is not probable that many bibliophiles in the domain of mathematics, seeing upon their shelves the sumptuous tall copies of the Opera Mathematica* of John Wallis, and consulting their noteworthy historical chapters, the first serious effort in the history of mathematics in England, ever consider that the author was one of the world's greatest decipherers of cryptic writing. To be sure his biographies give us the information that he was interested in cryptography, but the extent of this interest, the sixty years devoted to the subject, the services rendered to the State, the rewards and disappointments that came to him as a result—of all this the biographies tell us practically nothing. It is partly because of this fact, partly because of the interest naturally excited by the present war, but chiefly because of the new light that certain letters, hitherto unpublished, throw upon the life and character of a mathematician of merit, that this article is written.

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* Oxoniæ, MDCXCV.

After Wallis (1616–1703) had received the bachelor's degree at Emmanuel College in 1637 and the master's degree three years later, he became chaplain (1642) to the widowed Lady Vere. While so engaged there was brought to him a letter in cypher relating to the capture of Chichester.* In two hours he succeeded in deciphering it, and thus began his career as an expert in cryptography. As a reward for his services in this line he was given (1643) the living of St. Gabriel in London, and later (1647) that of St. Martin. His Savilian professorship (1649), his office of Keeper of the University Archives, and his chaplaincy to the king (1660), did not interfere with his work in the field of secret writing, and such was his standing that he was employed by the Earl of Nottingham as decipherer for William III. Loyal to his country, he steadily refused the importunities of Leibnitz to reveal his methods, but he made them known to his own son,† who lacked, however, the father's genius in this as in other respects, and to his grandson, William Blencow.‡

Some years ago, in working among the early mathematical manuscripts in the British Museum, I came across a considerable number of letters written by Wallis, many of them relating to this work in cryptography. While it is not feasible to publish them all, it may not be without interest to give a few of those which reveal the character of Wallis the man as well as of Wallis the cryptographer.§

The loyalty of Wallis to his own country, and his desire to have the work which he had begun for the state continue after his death, are seen in the following letter:

For Mr Tilson, at

Sir, Oxford, Mar. 20, 1700/1.

I am obliged to you (though unknown) for your favour in drawing up the Warrant to be signed by his

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* Dec. 27, 1642.

† John Wallis, b. Dec. 26, 1650, B.A., of Trinity College, Oxford (1669), called to the bar in 1676.

‡ He was the son of Sir John Blencow (b. 1642) who married the daughter of Wallis in 1675.

§ These letters are in Add. MS. 32499, entitled "Letter Book of Dr. John Wallis, 1651–1701."

Majesty for a Pension of 100$^{\text{lb.}}$ pann$^{\text{m}}$. payable to me, in order to my instructing a young man (William Blencow, son of Mr Justice Blencow) in the Art of Deciphering.

Concerning which I have this further to suggest; viz. That though it was his Mat$^{\text{ies}}$ chief design in that grant, partly to gratify me for services allready done, and partly to incourage my propagating of that Art, that it do not dy with me; I think yet he may be inclinable allso to incourage the young man to diligence therein; and perhaps may be gratiously pleased to order the pension payable to me during my life, (which is not like to be long (being now in my 85th year), and thenceforth to the young man during his Majesties pleasure.

I would have suggested this to my Lord Godolphin, or S$^{\text{r}}$ Stephen Fox, but that I have not the honour to be personally acquainted with either of them. But if this may not be obtained, then to have it pass as it is now drawn.

I may add this allso; That I have been solicited by Mijn Heer Leibnitz, more than once, in behalf of the Elector of Hanover: who is willing to send hither some young men, whom he desires I would instruct therein; leaving it to me to make my own proposals on what terms I would undertake it. To which I have returned answere, That I shal be ready, my selfe to serve his Electoral Highness if there be occasion: but the skill of doing it, being a curiosity which may be of use to my own Prince, I do not think it proper to send it abroad, without his Ma$^{\text{ties}}$ leave.

If you shal think it proper (in such way as you may judge most fit) that this be layd before his Ma$^{\text{tie}}$, you may therein oblige,

S$^{\text{r}}$ Your humble servant

JOHN WALLIS.

Of his painstaking work in deciphering, the following letter addressed to R. Hampden* gives evidence:

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* Richard Hampden (1631–1695) was a son of John Hampden (1594–1643) of revolutionary fame. At this time he was representing Wendover and was prominent as a spokesman of the Whigs.

For R. HAMPDEN ESQ^{r}, at the house of Comons.

S^{r} OXFORD, Aug. 3. 1689.

In a pacquet from the Earl of Nottingham, I lately received inclosed a Letter from you to which (in a like pacquet to his Lo^{p}. by the last post) I returned a short answere. But should have been more particular were it not for fear of loosing that Post; which was within a few minutes of going before I had dispatched that pacquet as it was; (though I had taken true pains about it, both by day, & by night, because you desired to have it done speedyly).

I find, in the Cipher, there had been some mistakes committed: As (in one place) *Brigadies* for *Brigadier* (where 36, had been miswritten for 35;) and *efget* for *effet* (twice;) and *pued* for *pied;* and *alller* for *aller:* All which (without altering the original, which I was not to do) I tooke the liberty to correct in my transcript. Beside which, I do not know that I have varied a letter from what was in Cypher, save that in one place, I doubt I have miswritten *pouroit* for *pouvoit*.

Some few words I could onely supply from the sense; As (in the second or third line) *c . . . . s*, of which I was sure (that is, that it was a word which begins with *c*, to which *s* was added in the end) for which (from y^{e} sense), I supply *chances* (because I could not think of any other such word which would make good sense, the same figure occurs not again.) In like manner *sans*, *tout*, *homme*, (& some few other) but these are confirmed by their so occurring several times in different circumstances; of which therefore I make no doubt.

For some few (of which I had some doubt) I have left blanks; as (in the second line) *entraine t . . . . . .* (where I think *train* or *traict* or some such word is to be supplied.*

Soon after, there is *c . . . . toute sorte de r . . . .* which is I presume to be read *contre toute sorte de regle:* which occurs again in y^{e}. next line, where *regle* is spelled by letters; unless in the former place we choose to read *raison*.

Some other French words I mett with, which I did not understand but I give them as I find them: & in some places there seems to me some (sic) a word wanting (which yet I would not take upon me to supply) as, where *donner* doth (two or three times) occur for *ordonner;* (to *order* or *give order*), but perhaps it is onely my being unacquainted with y.^{e} French way of speaking.

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* No; but tousiours [so in margin of original].

M^r d'Hamilton and *M^r Rose,* I do not know whether I am to read *Monsieur* or *Monseigneur* (because I do not know the quality of the persons;) But I think y^e. latter; because I find the same title afterwards given to the person to whom he writes, w^{ch} (as appears by y^e beginning & end of the letter) must there be read *Monseigneur.* But *M. . . . Melfort* (where another figure is used) is, I presume, to be read *My Lord Melfort.* But I chose to write *M^r d'Hamilton* &c because in the beginning of the letter (which is not in cypher) I find written *M^r d'Hamilton.*

Non estoit parfaitement b. . . . I suppose is to be supplyed by *bas* or some other such word beginning with *b.*

En e. . . . de tout craindre, is to be supplied by *exces,* or some such word beginning with *e.*

Ny a. . . . pas ici, is to be supplyed *avons.*

L'escrit m. . . . apres is to be supplied by *mesme,* or some other word beginning with *m.*

And, near the end, *e. . . . e,* is certainly to be read *envoyé,* but the word not then coming to mind, I leaft a blank.

Of all the rest, I think myself very secure not to have missed any thing.

When he says, there were killed ore wounded, *about fourty officers & some sixty soldiers;* it seems strange that, if 40 officers, there should not be more than 60 soldiers; unless *soixante* (60) be miswritten for *six cents* (600;) but I give you as it is.

The name subscribed, I know not wheither it (sic) *Mointis* or *Nointis* or *Pointis;* For it is so written as it may bear any of them. I take it to be *Nointis.* But it is surely the commander in chief there.

Nor do I know to whom it is written (because the cover was not sent me,) but it must be somebody much superior to him that wrote it; as appears by the manner of writing.

Nor know I how it becomes to be so rotten and discoloured in so short a time: Unless possibly it may have been thrown overboard into salt water & recovered from thence.

I know not whether it may not be thought fit to have it translated into English & printed.

S^r. you will please to excuse my giving you. the trouble of so long a letter about such little things. But I was willing to let you see how far I could conclude certainly, & where but by conjecture; & upon what kind of grounds I did so conjecture. Those perhaps who are better acquainted with the language, may, in these cases, conjecture better, than

S^r

Your very humble servant

JOHN WALLIS.

Some idea of the importance of the communications that came into his hands may be obtained from a letter addressed to the Earl of Nottingham, then one of the secretaries of state with charge of the war department.[*] The letter is as follows:

For the EARL OF NOTTINGHAM at Whitehal.

My Lord, OXFORD, Aug. 18, 1689.

I am allmost ashamed to tell yo^r. Lordship how much time & pains & study I have employed on that very perplexed cypher in the Letter from Poland, & have not yet dispatched it. But by what I have done all ready, I find two things (which seem to me) of moment. One is a Treaty (or intreaty rather) of the French King with the King of Poland presently to make a war on Prussia.[†] The other, about a marriage of the Princess of Hanover with the Prince of Poland, promoted by the French King. How far it may be of concernment to us to know it, I am no competent Judge; But I had thought it did become me to give this timely notice of it (lest there might be a prejudice by delay) while I am preparing to give a fuller account of that letter (than yet I can), by the next Post (or that next after it) if I have not in the mean time order to come up with it myself. who am

My Lord Your honours very humble servant

JOHN WALLIS

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[*] This was the second Earl of Nottingham (1647–1730), a great Tory but an anti-Jacobite.

[†] To give a diversion to the Emperour. [So in the original, as a note at the end of the letter.]

Many of the letters show Wallis as a man willing to serve the King but always seeking for rewards for himself and all his family. The following gives some idea of his methods of ingratiating himself with the powers at Court, a procedure then undoubtedly less objectionable in the eyes of the world than now:

To y^{e} EARL OF NOTTINGHAM at Whitehall.
My Lord, OXFORD, Nov. 12, 1689.

I have ordered my son (who is now, or will be sudainly in London,) to wait on your Honour & acquaint you that I sent you a packet by last Sunday's Post; & two before by Sunday & Tuesday Posts next foregoing; which I hope came safe to yo^{r} Lo.^{ps} hand. But I am now at a stand, by reason that the five other letters are at lest in three (I think in four) different ciphers, & all new, which I have not had before. Besides which, I have myself (since my last) been very ill, so as not to be in a capacity to apply myself (as I would) to that or any other business which requires intentness; & continue stil indisposed. But I shal (as my strength & y^{e} difficulty of y^{e} work will permit) endeavour to serve his Majesty, & your Lordship therein.

I humbly thank yo^{r} Lordship for the kind expressions (of your readyness to oblige me) in yo.^{r} Lo.^{ps} letter of Octob. 24. I have not been fond of being a great man (studying more to be serviceable than to be great) & therefore have not sought after it. Especially in the late times, wherein I could safely foresee that divers things might then be expected from me, with which I could not comply. But I am capable of any promotion Ecclesiastical, or in a way of literature; & of my accomplishments for either, I had rather others should speak than myself. And I have a son, a Barrister of the Inner Temple who though his attendance on an estate he hath in the country have caused him to neglect the practise of y^{e} law. at London, yet he understands business very well, & is sagacious inough in it, & capable of any civil imployment; & of whom I could say more if he were not my son. And I hav a son-in-law (who married one of my daughters) M^{r}. Serjeant Blencow (of y^{e}. Inner Temple allso) an able Lawyer both at Chamber practise & the barr; & not inferior therein to divers of those on the Bench; And neither of them in the lest wise sullyed by any unhandsome actings in the late bad times; but are very cordial to his Ma^{ties} interest. Any thing of kindness to either of them, will be so, to

My Lord, Yo^{r}. Honours very humble servant

JOHN WALLIS.

Another letter, also showing the desire to be in the favor of the Court, and having some interest in view of the present European situation, is as follows:

To y^{e} EARL OF NOTTINGHAM at Whitehal.

My Lord, OXFORD Febr. 20, 1689/90. Thursday.

Before I left London (which was on Munday last) I was (on Saturday) to wait on yo^{r}. Honour both at Berkshire-house, & at y^{e}. Office in Whitehal, to receive your commands for Oxford, but was not so happy as to find you at either. And I had y^{e} same day waited on the Envoy of Brandenburg on the like occasion. Some while before which, a friend of mine dining at the Table of a Noble Lord, where divers Lords were present, & other persons of quality, one of them did openly declare, that y^{e} Elector of Brandenburg had sent me, for a present, a large medal, with an Inscription & a chain of gold (for deciphering French Letters &c). My friend sayd, He thought it was a mistake; for that he had spoken with me the night before, & I knew of no such thing. The other person of quality confirmed what he had before sayd, & went on to describe it so particularly as if he had seen it. Now (beside that this was not so) I doubt whether, if it were, it were advisable to make the thing so publike. That it was not so, I find, because I have since waited on that Envoye, & he sayd nothing to me to that purpose. He sayd indeed that his Master intended to send me a mark of his favour; but that none such was yet come; & added withal, that it is as the King's concern to gratify me for those services, as being a joint interest (sic) of him & his Allyes. And, why I doubt whether it be advisable to make it so publike (that I decipher y^{e} French Kings letters) is, because it can then hardly be avoided but that this will some way or other come to y^{e}. French Kings knowledge from some of his correspondents here; which will be attended at lest with one or both of these inconveniences, viz: A greater care to prevent the intercepting of such letters; And, a change of the ciphers they now use (which they have allready changed more than once, for I have allready nine or ten of their ciphers by mee) for others more difficult, & (perhaps) in superable. The two letters which I have brought with me from London, (one of two sheets, the other of five sheets) I shall endeavour to dispatch as soon as I get a little leisure. And am

My Lord, Yor. honours very humble servant
JOHN WALLIS.

That Wallis often felt that the rewards were not commensurate with the labor, and that he did not hesitate to make this fact known, may be seen from a letter written in 1691:

For Mr HARBORD, in St. James's park London.
Sir, Aug. 15, 1691.

I should begin my letter with thanks for yor civilities of late, & before; And that you have pleased to concern yourself for me without my knowledge, (and without any complaint of mine.) But (because your time is precious) I shall rather obey yor commands, in giving you some account of what you ask me.

It is now more than two years compleat, since by a letter from my Lord Nottingham & another from Mr Hampden, I was desired to decipher an intercepted French letter, from ye camp then before London-Derry; and (soon after) a second from ye same place; &, then, a third from Poland to the French King from his Ambassador there. Which I performed to that satisfaction, yt his Lordship made me a Present (from ye King, I suppose) of Fifty pound. Which I looked upon as a handsome gratuity for ye service then done, & as a testimon of his Maties acceptance (which I valued) & returned my acknowledgements accordingly.

After this I was kept in a continual imployment of like services: wherein I took true pains, (& my son with me, who was assistant to me;) by day & by night, (because letters came faster than we could well dispatch them:) And it was almost our whole imployment for a long time.

About a year after; having received nothing (but good words) in requital (since y^t for y^e first three letters) I let M^r Hampden know so much (upon who's joint desire I had undertaken y^e service.) Upon who's speaking(I suppose) to my Lo. Nottingham, he sent me another 50^l, which is all I have received.

Perhaps you will judge this to be no great gratuity for so much service (which every body could not have done him:) Having by that time deciphered some hundred sheets of paper, sent me in cipher, for that purpose. And some of them, (he can tell you,) were of very great concernment: The deciphering some of those letters having quite broke all y^e French King's measures in Poland for that time; & caused his Ambassadors to be thence thrust out with disgrace. Which one thing was of much greater advantage to his Ma^{tie} & his Allies, than all that I am like to receive on that account.

I remember that my Lord Arlington (a little before he gave over being Secretary) sent for me on purpose, & did (without my asking) give me 50 Guinny's in hand; & promised me 50 guinney's a quarter (which were duely payd me) to endeavour the deciphering of such letters as should be sent me from time to time. And I had of him (as I remember) 200 Guinney's before he gave over y^e place of secretary. Yet I had not done for my Lord Arlington the tenth part of that service I have done my Lord Nottingham.

However; without disputing the point, whether that Fifty pound were a Noble recompense for all y^e services then done (except y^e three first letters) & for all that I was to do afterward (for that, it seems, is now the case:) I have continued the same trade for more than another full year, without having received (or being like to receive, for ought I see) y^e value of one penny.

'Tis true; I have had all along a great many good words; That he is my humble servant; My faithful servant; my very faithful servant; That he will not fail to acquaint y^e King with my diligence & success in this difficult matter; That he is very sensible of y^e pains I had taken in this troublesome work; That he shal be very glad of any opportunity of doing me any service: That y^e King is sensible of my zeal & good affection, and will (he beleeves) in a short time give me some mark of his favour (meaning, I suppose, some preferment,) wherein his Lo^{ps}. endeavour shal not be wanting; That he will not be wanting to represent my services to my advantage. And y^e like.

But these Promises are of Two-years standing. And it may seem strange; if in all that time (while so many Deans, & Bishops, & Arch Bishops have been made) his Lo^p. could never find an opportunity of what he so much desired.

Now, y^e Apothecaries tell us, that some Cordials are y^e less Operative, for having been long made. And I have been told (by one who pretends to understand them) that Courtiers promises, if they do not operate quickly, it is not likely y^t they will operate at all, (but are to pass for words of course:) and that to depend upon stale promises, is like calling for an Old Debt; w^{ch} y^e longer it hath been forborn, grows y^e more desperate.

As to what he intimates of preferment: I have asked none, & they offer none, (& so far we are are agreed:) And there are so many who (by themselves or friends) seek preferment, y^t it is not like to fall upon one who looks not after it. And I easyly foresee so many exceptions, & Diversions, that may be made by persons otherwise concerned, (as, y^t one thing is too good for me; another, not good inough; a third, fit for somebody else, y^t is to be first served: and y^e like:) that I have no great expectation of it (being not acquainted with y^e methods of Court applications) nor much concerned for it: Having allways endeavoured rather to deserve preferment, than to have it: I may perhaps, in time, be offered something that may not be advisable for me to embrace, (because I must, for it, leave what I have in Oxford; which if not better than it; may at least be so to me;) & then it will be easy to object that he who doth not accept a little one, doth not deserve a better.

But having (for my Lord Nottingham) condescended to do clarks-work; I might at least expect clarks-wages; (without being thought mercenary or ungentile:) And I presume there is never a clark his Lo^p. keeps, but is (one way or other) better payd, for y^e work he doth, than I am.

He may say perhaps, This is (not his, but) the King's service. Very true. And so is all y^e service his Lo^p. doth as Secretary. Yet he is well payd for it. And, so wel, that he may (out of his allowance) afford to gratify those y^t work under him.

All which is sayd, not that I do distrust their Ma^{ties} gratious inclinations; or, would or (sic) under value their favours. But, because Princes see with other mens eyes; & know no more of Persons or services than as they be represented. And then, if one man (by a friend at Court, who hath the Kings ear) be represented as a person of prodigious parts, of great learning, of a clear reputation, of great esteem at home & abroad (in all parts of y^e learned world,) as a credit to y^e University, an honour to the Nation; as a friend to the Government, & zealous for it; of a good judgement, a clear apprehension; that speaks well & writes well, in what he undertakes, & may (if imployed) be many ways serviceable to the Government; a man of temper & moderation: not bigotted to a faction (laying great weight upon little things, which cannot bear it, or deserve it;) who hath done, & can do services which others do not pretend to: (though there may be much of Hyperbole in all this:) And another be represented, as, An obscure person (not known at Court,) that hath a knack perhaps to decipher a letter, or so, (& may be made use of for such a purpose,) but makes no figure in the world, nor looks for any great matters; & may be easyly induced (by a few good words) to do what you would have him: In such case, I say, it is not hard to judge, whether of the two stands fairest to be preferred.

Nor is it sayd, as if I were not willing to serve their Ma^{ties} without being Hired. But the quite contrary. I have been, & shal be willing to serve their Ma^{ties} the best I can upon all occasions. And to do it gratis (if it be their pleasure it should be so:) & do it as honestly as those who are better payd. Of their Ma^{ties} I make no complaint at all. All we can do for them is little inough: & it is impossible they can gratify all.

But when I am imployed (in troublesome work) by persons of Honour: I expect they should do honourable things: And, if therein I am mistaken, it is their fault, not mine. I am

S^r your very humble servant

JOHN WALLIS.

Out of the mass of letters on the subject, of which I have copies, a single example may be given in conclusion to show both certain mental characteristics of Wallis and also something of his elementary work with ciphers. The letter was written to James Johnston* whom Swift called "one of the greatest knaves, even in Scotland."

> To M^r JOHNSTON Ambassador to Berlin.
> S^r June 9, 1692. SOUNDESS BY NETTLEBED,
> OXFORDSHIRE.
>
> Your letter to me of May 28. came to Oxford at a time when I was not there; nor have I been there since, but it was sent after me where I now am. I thank you for the kindness you did me, or would have done me, at Berlin. And if your endeavours did not succeed, I am not to blame you for it. Onely I could have wished, you had told y^e Elector himself that I had yet received nothing (& he might then use his discretion,) rather than to Mons^r Smetteau who knew it well before. Of whom you will give me leave to say, that he might perhaps act like a Statesman, but certainly not like a Gentleman; to treat me like a child, as if I were to be wheedled on to difficult services by a few fair words, & a promise of sugar-plums, which should in the issue signify nothing. I did not want his present, (& perhaps can spare it as well as he,) nor did I ask any; till he told me of his own accord that his Master took my service very kindly, that he intended me an honorable present, & had given him orders to tell me so, & he expected it by y^e next Post. Nor did he onely say so to me, but caused it to be spread abroad (for I heard it from divers hands) that y^e Present was come, that it was a rich medal with an honorable inscription, & a gold chain of a great value. Now if such present was sent me, but never came to my hand; I was wronged on a double account: first in missing the present that was intended me; & then in being thought Rude by his Electoral Highness for not returning some acknowledgement for the bounty which he thought I had received. But if no such present was sent, nor was intended; He should have sayd nothing of it, & I should have sayd as little. Yet when

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* James Johnston (1643?–1737), son of Sir Archibald who was hanged for treason (1663) by order of Charles II.

I had deciphered for them between two & three hundred sheets of very difficult, & very different cipher, they might at lest have afforded me Porters pay; if not that of a Scrivener. Yet I would not be thought in this to reflect on his Electoral Highness, but on Myn-heer Smetteau. For Princes know no more of Things or Persons, in such cases, but as they be represented by their Ministers. His last evasion, of y^{e} Coyners slowness, is so pittyfull an excuse that I am ashamed to hear it. For if a Coyner cannot, in three years, prepare a Medal; a Goldsmith certainly might, in a less time make a Gold-chain; (or somewhat equivalent:) or if he had remitted hither the value in Mony, it might have been made at London. But inough of this.

As to what you desire, concerning an easy Cipher, which yet may be tolerably safe, because you are to use it yourself & cannot well spare much time for it: I think you may venture on such as this,

For y^{e} letters, a b c d e f g h i k l m n o p q r s t u
w x y z, put the next after them, b c d e f g h i k l m
n o p q r s t u w x y z a, or the next before them, z a b c
d e f g h i k l m n o p q r s t u w x y,

or the next but one, or two, before or after them in the Alphabet; and intermix (as Nulls) the numeral figures, 1 2 3 4 5 6 7 8 9 or the greek letters α β γ δ ϵ ζ &c, or what other such notes you please. (And you may by one such alphabet correspond with one person, & by another with another whereof neither shal understand the other's cipher. (Which is a method very ready for use; (And I know that K. James with his correspondents here doth use Ciphers not better than these.)) And 'tis tolerably safe, on these accounts. 1. 'Tis a chance, whether or no the letter be intercepted. 2. If so, 'tis yet a further chance whether those who have it will attempt its being deciphered (& not rather neglect it as thinking it to no purpose.) 3. 'Tis a further chance whether they light upon one who can do it. 4. If deciphered 'tis yet a chance whether it do you any considerable mischief; (for I presume you will not intrust any thing of very great moment therein.) And of all these casualties if any one so happen as you would have them, you are safe. If you desire one which (with a little more trouble) may be somewhat more safe, I may send you one from Oxford (whither I am now going) from

S^r your humble servant,

These, JOHN WALLIS.

For M^r J. JOHNSTON, Secretary for Scotland;
at my Lord Nottingham's Office, at Whitehall, London.

SHORTER NOTICES.

Analytic Geometry and Calculus. By FREDERICK S. WOODS and FREDERICK H. BAILEY. Ginn and Company, 1917. 516 pp.

THIS book is a revision and abridgment of the authors' Course in Mathematics for Students of Engineering and Applied Science. In making this abridgment the authors have omitted determinants, theory of equations, poles and polars, diameters, center of curvature, special methods of integration, and complex numbers.

The first eight chapters deal entirely with analytic geometry and give the subjects usually given in a first course in American colleges. The following topics are exceptionally well treated: "Variables and functions," "Graphs" and the derivations of the standard equations of the conics. In several places the authors are very careless about a theorem and its converse, i. e., they prove a theorem and then state or use its converse. An example of this is on page 61 where they prove that two perpendicular lines have their slopes negative reciprocals of each other and then conclude that "two lines are perpendicular when the slope of one is the negative reciprocal of the other." Moreover no attention is called to the fact that if the lines are perpendicular to the coordinate axes their slopes are not negative reciprocals.

In Chapter IX they introduce calculus by means of slope and area. This is a very well written introduction to the calculus except for the fact that the definition of limit on page 130 is incorrect (the word "numerically" should be inserted after the word "remains" on line 4). Then follow chapters on the conventional work of maximum and minimum, tangents and normals, integration, etc. This is followed by Chapter XIII on Applications of integration which is exceptionally well written. The fundamental idea of the summation process of the integral calculus is given in the clearest manner. On page 268, line 10, a lower limit has been omitted and at the bottom of the same page the letter A stands for a point, while on the next page (line 8) it mysteriously becomes an area.

Passing from the best-written chapter in the book we come to the poorest, namely Chapter XIV, on Space geometry. In this chapter the authors have made many inaccurate statements. For example on page 310 we find: "Find the equation of the sphere formed by revolving the circle $x^2 + z^2 = a^2$ about OX as an axis." Now $x^2 + z^2 = a^2$ represents a cylinder and not a circle. The authors evidently meant the circle $x^2 + z^2 = a^2$, $y = 0$. The proof of the distance formula from a plane to a point is also incorrect, for it assumes that the plane cuts the z-axis. The same trouble exists on page 63 in the proof for the distance formula from a line to a point; there it is assumed that the line cuts the y-axis.

The chapter on space geometry is followed by chapters on Partial differentiation, Multiple integrals, Infinite series and a short course in Differential equations.

Many fine examples are worked out in the text and many more are given in the exercises, the total number of which is 2,000. These exercises are to be found only in long lists placed at the end of each chapter. The book is well adapted for use in a course covering both analytics and calculus.

F. M. MORGAN.

Introduction to the Calculus of Variations. By W. E. BYERLY. Cambridge, Harvard University Press, 1917. 8vo. 48 pp. Cloth. Price 75 cents.

THIS little book is the first of a series of "Mathematical Tracts for Physicists." It indicates in admirably clear style the solution of a number of examples involving some of the fundamental ideas of the calculus of variations. As the subject owed its origin to the attempt to solve a rather narrow class of problems in maxima and minima, the eight pages of the Introduction are mainly taken up with a discussion of three simple examples: the shortest line, the curve of quickest descent, the minimum surface of revolution.

The integrals of the Lagrange equations arising in connection with the second and third of these examples are given in the second chapter (pages 9–22) which is entitled: "Variations. Notation and nomenclature. Illustrative problems." We find here also a second solution of the two-dimensional shortest line problem (polar coordinates), and a solution of the problem of the geodesic line joining two given points on the surface of a sphere. Section 10 is devoted to isoperimetrical problems. The seven examples that are given for solution involve slight developments of the text.

In Chapter III (pages 23–28), on "Problems involving several dependent variables," Hamilton's principle and its application are considered. Chapter IV (pages 29–33) on "Multiple integrals" contains (1) the derivation of the differential equation of minimal surfaces, and, by means of Hamilton's principle, (2) the derivation of the differential equation for small transverse oscillations of a stretched elastic string.

"Variation of limits" and the "Principle of least action" are the topics of the last chapter. In each of the last three chapters are examples to be solved.

No references to the literature of the subject are to be found in the tract.

R. C. ARCHIBALD.

Science and the Nation. Essays by Cambridge Graduates with an Introduction by the Right Hon. Lord Moulton. Edited by A. C. SEWARD. Cambridge, at the University Press, 1917. 22 + 328 pages. Price 5 shillings. (G. P. Putnam's Sons, New York. Price $1.50.)

"IT is widely believed that technical education stands for efficiency and prosperity, but pure science is regarded as something apart—a purely academic subject. It was with a view to demonstrate the fallacy of this distinction that the present volume was suggested," writes Mr. Seward.

The volume contains thirteen chapters each written by a specialist of note. The first four chapters, occupying about one third of the volume, are headed as follows: "The national importance of chemistry" by W. J. Pope; "Physical research and the way of its application" by W. H. Bragg; "The modern science of metals, pure and applied" by W. Rosenhain; "Mathematics in relation to pure and applied science" by E. W. Hobson.

Some of the topics in Professor Hobson's very interesting 27-page chapter are: mathematics and practical life; evolution of mathematical concepts; physics and mathematics; dynamical theory of tides; mathematics and physical science; optics and mathematics; abstract dynamics; geometry; reform of teaching; the teaching of mechanics; university teaching; research.

The volume as a whole is exceedingly readable and the reviewer considers that a perusal of its pages must convince any one that pure science is *not* to be regarded "as something apart."

R. C. ARCHIBALD.

Cours de Mécanique. Vol. II. By LÉON LECORNU. Paris, Gauthier-Villars, 1915. 538 pp.

THIS is the second of the three volumes of a treatise on mechanics for use in l'Ecole Polytechnique. The first volume was reviewed in the BULLETIN for April, 1915. The present volume is devoted to theoretical mechanics with a few immediate applications and it is the intention of the author to treat the mechanics of engineering in the third volume.

The first four parts of the complete course are contained in the first volume. The second volume begins with the fifth part, on general dynamics, which includes the fundamental theorems on the motion of a system of particles, the theory of energy, constrained motion, impact, the equations of Lagrange, and small oscillations.

The dynamics of a rigid body is the subject of the sixth part and includes the following topics: moments of inertia, rotation about a fixed axis, motion parallel to a fixed plane, motion about a fixed point, motion of a heavy homogeneous solid of revolution on a smooth horizontal plane, motion of a solid subject to no restraints, and a brief chapter on an extension to the motion of a deformable system.

The seventh part, on the dynamics of physical solids, is devoted principally to a discussion of friction. The theory of elasticity is given in the eighth part, and the volume is concluded by the ninth part on the mechanics of fluids.

W. R. LONGLEY.

Les Théories statistiques en Thermodynamique. Par H. A. LORENTZ. Leipzig, B. G. Teubner,.1916. 120 pp.

DELIVERED at the Collège de France by Lorentz in 1912, reported and edited by L. Dunoyer in 1913, and finally printed by the author in 1916 with a mathematical appendix supplementing the text, these lectures may be characterized as a frank discussion of some of the most troublesome points in recent thermodynamical theories. Lorentz never throws dust in the reader's eyes; whatever is doubtful is plainly so labeled; that which is tentative is distinguished from that which is settled; the physical ideas are discussed with a minimum of mathematics, and all extended analytical work is reserved for the appendix, where it can neither befog nor interrupt the even tenor of the work.

The relation of entropy to probability in classical thermodynamics, with applications to monatomic gases and other substances, the effect of the size of molecules, the comparison of two possible definitions of probability (Boltzmann and Einstein), canonical ensembles (Gibbs), and fluctuations in statistical phenomena are treated in the first three lectures. The last topic is of great importance and of recent development, with applications to the emission of α particles, to the scattering of light, to Brownian motions, and to black body radiation (fourth lecture). The initial impulse in this field has been largely due to Einstein. Lorentz's connected exposition will be welcomed. A treatment of Planck's formula and of the theory of quanta, admirably honest, closes the set of five suggestive lectures. There is philosophy as well as physics in the work.

There is no other text which can be recommended as comparable with this for clarifying ideas in a very opaque field, and no great knowledge of mathematics or physics is a necessary preparation for enjoying the book, for sensing the exhilarating boldness of some recent theories, and for discerning possible directions of escape from present perplexities.

E. B. WILSON.

The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat. By H. A. LORENTZ. Second Edition, Leipzig, B. G. Teubner, 1916. 343 pp.

THIS second edition of Lorentz's Theory of Electrons is practically identical with the first edition, except for a few changes in the footnotes and in the mathematical appendices or notes, which follow the main text. For the most part these changes are for the purpose of giving a clearer statement of the theory of relativity and of the theory of quanta, subjects which were not as prominent in 1909 as they are at present. Except in these two matters, the theory of electrons has not progressed much in the last half dozen years. I need only refer, therefore, to my review of the first edition, BULLETIN, volume 17, pages 194–200.

E. B. WILSON.

Einführung in die theoretische Physik in zwei Bänden. Von CLEMENS SCHAEFER. Bd. I. *Mechanik materieller Punkte, Mechanik starrer Körper und Mechanik der Kontinua (Elastizität und Hydrodynamik).* Leipzig, Veit, 1914. xii + 925 pp. + 249 fig.

THE author, Dr. Schaefer, is well known for his popular and excellent introduction to Maxwell's theory. It was the success of this work, with its clear indication of Schaefer's ability as a writer, that led his friends to press him to publish an introduction to theoretical physics—a large work intended to cover a course of lectures four times a week for five or six semesters. There are not so many general treatises on theoretical physics of the scope of this one that the author need fear the chance of duplicating existing texts.

The present volume deals with mechanics taken in a broad way, covering the mechanics of a particle, of systems of particles, of rigid bodies, and of continua; that is, of fluids and elastic bodies.

The treatment of mechanics of a particle contains chapters upon kinematics, general principles of dynamics, special cases of motion (particularly oscillatory and including Foucault's experiments), general principles of dynamics of systems of particles and their application to special systems. As might be expected, the material does not differ very greatly in sort or in treatment from that found in treatises on the mechanics of a particle. The introduction of Coriolis's theorem on relative motion does, however, come somewhat earlier than usual. A small amount of vector analysis is used, being developed from time to time as it is needed. Moreover, there is a good and unusual section on free and forced oscillations of finite amplitude, which has interesting applications in a theory of sound, and there is a demonstration of Dirichlet that in a statical system the potential energy is at minimum for stable equilibrium—a fact which is often assumed without demonstration. Hamilton's principle and Lagrange's equations are also taken up before the theory of the motion of a system of particles is left.

The mechanics of rigid bodies starts with kinematics, in which there is given a considerable discussion of the geometry of rigid motion. The dynamical part is founded upon d'Alembert's principle, and in treating the momental ellipsoid a certain amount of the theory of the linear vector function is presented. The general theory of the dynamics of rigid bodies concludes with a section on statics and it may be well to point out that the author follows modern procedure in making statics depend upon dynamics. There is a considerable treatment of the theory of the top and of potential theory, including Poisson's equation.

The treatment of continua begins with the theory of infinitesimal displacement, of which the treatment is detailed even to the introduction of the divergence and curl. Dynamics, when applied to continuous media, begins with statics,— a procedure just the reverse of that followed in treating rigid bodies. We may well ask whether it is logical to indulge in this chiastic arrangement, and are we likely to maintain it, or shall we ultimately come to begin the theory of continua with kinetics, or may we revert to the old-fashioned method of putting statics everywhere before kinetics. After treating various cases of elastic equilibrium, the author takes up the motion of an infinite medium (wave motion) and follows it with the oscillations of strings and membranes; and here we find Green's functions and integral equations, which we should not have found, at any rate in the same form, in older texts, although Lord Rayleigh and Oliver Heaviside long ago used methods not wholly dissimilar. In more or less parallel fashion the author then takes up the oscillation of bars and plates.

Finally he goes on to hydromechanics, starting with equilibrium and with wave motion. He follows this with a chapter upon motion with a velocity potential including a treatment of Stokes's theorem. He gives particular attention, as is customary, to two-dimensional motion and develops a few cases of discontinuous motion wherein the theory of functions of a complex variable is used. The fundamental properties of vortex motion are well treated, and finally the theory of friction in liquids, which closes with a few words on turbulence and on Prandtl's investigations.

This rapid sketch of the contents of Schaefer's first volume will indicate that the method followed by the author is for the most part to treat fundamental theorems in detail and we may add with clearness, to give a few of the most important applications of the theorems, but to leave aside anything that would be considered as distinctly special and better suited to technical texts on the various branches of mechanics.

Here in America the work can scarcely be used otherwise than as a book of reference. Few universities, if any, offer a course, or are in a position to offer one, which should stretch an introduction to theoretical physics over five or six semesters with four lectures per week. Those of us who give courses on the subject generally find it better to proceed in a very different way; namely, to give in one course perhaps of three hours per week for a year of thirty weeks, a general introduction to mechanics, heat, light, and electricity, leaving the student at the end of the course in a position to undertake serious courses in the various branches of physics, such as rigid mechanics, hydrodynamics, elasticity, optics, thermodynamics, or electricity and magnetism. For our conditions this is undoubtedly a better arrangement because, after the one introductory year, the student is free to turn his attention intensively to any one of the major branches of physics, whereas on the plan followed by Dr. Schaefer, he would be methodically carried through each of the branches in about three years; thus, so far as I can see, getting in no one of those branches the detailed treatment which he would get of one or two of them under our method.

In teaching theoretical physics, it is comparatively easy to give a set of reasonably satisfactory lectures, and it is equally difficult to gather together as satisfactory a set of exercises for the student to work. Unfortunately, though the lecturer may find much in this book which will aid him in the preparation of his lectures, the teacher will find little or nothing which will give him hints of material suitable for assignment as exercises for his pupils, and somehow or other it seems to be true that without exercises American students, even the most industrious, do not gain the grip upon a subject and the insight into it which they should. English books, to a great extent, and French treatises to an increasing extent, are setting forth exercises for the student to work, and it may be hoped that sooner or later this style of composition may appear in Germany. From some points of view, placing exercises in a book like Dr. Schaefer's might appear uncalled for, and he can hardly be blamed for omitting them, but it is not difficult to believe that even in Germany students would appreciate their presence.

EDWIN BIDWELL WILSON.

NOTES.

THE second summer meeting of the Mathematical Association of America was held at Western Reserve University and the Case School of Applied Science, Cleveland, Ohio, on Thursday and Friday, September 6–7, 1917, immediately following the summer meeting of the American Mathematical Society. The joint dinner of the two organizations took place on Wednesday evening, and a joint session was held on Thursday morning at which Professor L. P. EISENHART, of Princeton University, delivered by invitation an address on "Darboux's contribution to geometry." The further programme included papers by H. E. HAWKES: "Undergraduate mathematical clubs," with discussion led by R. C. ARCHIBALD and D. A. ROTHROCK; E. R. HEDRICK, retiring presidential address: "The significance of mathematics"; E. B. STOUFFER: "Geometry for juniors and seniors," with discussion led by ARNOLD EMCH and L. W. DOWLING; L. C. PLANT: "The treatment of the applications in college courses in mathematics," with discussion led by W. A. HURWITZ and A. M. KENYON. An informal report on the activities of the national committee on mathematical requirements was presented by the chairman, J. W. YOUNG; this report will be published in an early number of the *American Mathematical Monthly*. Ninety persons, representing fifty-one institutions, attended the various sessions. The Council elected nine individuals and two institutions to membership.

THE American association for the advancement of science will hold its seventieth meeting at Pittsburgh from December 28 to January 2 under the presidency of T. W. RICHARDS, of Harvard University. The retiring presidential address will be by C. R. VAN HISE, president of the University of Wisconsin. H. N. RUSSELL, of Princeton University, is vice-president and F. R. MOULTON, of the University of Chicago, is secretary of Section A.

THE September number (volume 19, number 1) of the *Annals of Mathematics* contains the following papers: "Minimal surfaces applicable to surfaces of revolution," by J. K. WHITTEMORE; "A reduction of certain differential equations of the first order," by W. D. MACMILLAN; "The function $W = L(\tau)$ defined as the inverse of the function $\tau = W - \log W$," by W. D. MACMILLAN; "Solution of the matrix equation $X^{-1}AX = N$," by H. T. BURGESS; "Linear functionals of N-spreads," by C. A. FISCHER; "Correction," by L. I. HEWES; "Substitution groups and possible arrangements of the players at card tournaments," by G. A. MILLER; "Problems of the calculus of variations invariant under a continuous group," by I. A. BARNETT; "A proof of Petersen's theorem," by H. R. BRAHANA; "Complete existential theory of Bernstein's set of four postulates for Boolean algebras," by J. S. TAYLOR; "Ratio, proportion and measurement in the elements of Euclid," by H. B. FINE.

THE English quarterly *Science Progress* has for the last two years published regularly a set of reviews, with full references to current scientific literature, of recent advances in science. With the first number of the twelfth volume (July, 1917) these reports have been given a more prominent place and increased in number and length. Those which are of the greatest interest to mathematicians are on philosophy, by H. ELLIOT; mathematics, by P. E. B. JOURDAIN; astronomy, by H. S. JONES; physics, by JAMES RICE; physical chemistry, by W. C. M. LEWIS; and crystallography, by ALEXANDER SCOTT. Of these the article on mathematics is the longest, and it is of interest to see accounts of recent work appearing much more promptly than is the case with the *Fortschritte* and the *Revue Semestrielle*. The reports are necessarily very short, but there is a welcome attempt at selection and criticism of important papers which marks a certain advance over the procedure of the older journals. It is especially interesting to observe the increasing part taken by America in the mathematical work of the world.

In *Notes and Queries*, London, for August, 1917, J. H. RIVETT-CARNAC has the following note on "Magic squares in India": "In the annual report for 1916 of the Archaeological superintendent of Hindu and Buddhist remains in northern India it is stated that amid the ruins of a temple at Dudhài, Jhansi, was to be seen a stone on which was engraved a magic square: (1) the sum of each row, each column, and each diagonal is 34; (2) the sum of all the numbers in each subsquare is 34. Such squares probably are well known, but to find one in a Hindu temple assigned to the eleventh century A. D. is unusual, perhaps. The figures are 1 to 16."

THE following is a partial list of university and college teachers of mathematics who have recently entered the military service:

Dr. J. W. ALEXANDER, Princeton University, second lieutenant, N. J. national guard. Professor A. A. BENNETT, University of Texas, captain, coast artillery. Professor A. D. BUTTERFIELD, Worcester Polytechnic Institute, captain, aviation branch of signal corps. Dr. J. F. CONNER, Catholic University of America, first lieutenant, bureau of supplies and accounts, navy. Dr. H. B. CURTIS, Columbia University, second lieutenant, aviation corps. Professor PETER FIELD, University of Michigan, captain, coast artillery. Dr. R. E. GILMAN, Cornell University, captain of artillery. Dr. W. L. HART, Harvard University, second lieutenant, field artillery. Mr. L. S. HILL, Princeton University, ensign, navy. Mr. J. W. HOPKINS, Harvard University, second lieutenant, heavy artillery. Professor S. D. KILLAM, University of Alberta, gunner, Canadian artillery. Dr. E. A. T. KIRCHER, Harvard University, captain, coast artillery. Dr. J. V. McKELVEY, Cornell University, second lieutenant, national army. Dr. T. A. PIERCE, Harvard University, second lieutenant, national army. Professor A. L. UNDERHILL, University of Minnesota, captain, coast artillery. Professor OSWALD VEBLEN, captain, ordnance, reserve corps. Professor K. P. WILLIAMS, Indiana University, captain, field artillery.

PROFESSOR PAUL PAINLEVÉ, of the University of Paris and recently minister of war, has been made premier of the French Republic.

AT Washington University, associate professor W. H.

Roever has been promoted to a full professorship of mathematics and appointed acting head of the department.

Professor P. P. Boyd, of the University of Kentucky, has been appointed dean of the college of arts and sciences and acting president of the university.

At Pennsylvania State College, associate professors J. H. Tudor and H. F. Stecker have been promoted to full professorships of mathematics.

Dr. R. A. Johnson, of Western Reserve University, has been appointed professor and head of the department of mathematics in Hamline College.

Dr. Ida Barney, of Smith College, has been appointed professor of mathematics at Lake Erie College, Painesville, Ohio.

Professor A. B. Frizell has resigned his professorship of mathematics in McPherson College.

Professor G. H. Scott, of Yankton College, has resigned to become principal of Benzonia Academy, Benzonia, Mich.

At the University of Chicago, assistant professor A. C. Lunn has been promoted to an associate professorship of mathematics.

At Northwestern University, assistant professor E. J. Moulton has been promoted to an associate professorship of mathematics. Mr. F. L. Kerr and Mr. A. D. Campbell have been appointed instructors in mathematics.

Dr. W. V. N. Garretson, of the University of Michigan, has been appointed associate professor of mathematics in Pennsylvania Military College.

Professor M. O. Tripp, of Olivet College, has been appointed to an assistant professorship of mathematics in the University of Maine.

At the University of Toronto, Dr. Samuel Beatty has been promoted to an assistant professorship of mathematics.

Dr. C. C. Crump, of Carthage College, has been appointed assistant professor of astronomy at Ohio Wesleyan University.

Mr. Herman Betz, of Cornell University, has been appointed instructor in mathematics in the University of Michigan.

Dr. W. W. Denton, of the University of Illinois, has been appointed instructor in mathematics in the Worcester Polytechnic Institute.

Professor C. A. Barnhardt, of Carthage College, will be a member of the mathematical faculty of Colorado College during the coming academic year.

Dr. W. H. Besant, fellow of St. John's College, Cambridge University, and university lecturer in mathematics, died June 2, 1917, at the age of eighty-eight years.

Professor S. B. Kelleher, of the University of Dublin, died August 18, 1917.

Baron Dairoku Kikuchi, formerly professor of mathematics in the University of Tokyo and at one time president of that university, died August 20, 1917. Among other offices held by him was that of minister of education and privy councillor. Baron Kikuchi was one of the first Japanese scholars to be educated in England and to introduce modern mathematics into the universities of Japan. He was well known for his contributions to the history of Japanese mathematics.

Mrs. E. B. Davis, of the Nautical Almanac Office, Washington, D. C., died in April, 1917. Mrs. Davis had been a member of the American Mathematical Society since 1910.

Book Catalogues:—Deighton, Bell and Company, Cambridge, England, catalogue 28, exact sciences, 1900 titles.—Dulau and Company, 37 Soho Square, London, catalogue 68, natural and mathematical sciences, 1286 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Abhandlungen über den mathematischen Unterricht in Deutschland, veranlasst durch die Internationale Mathematische Unterrichtskommission. Herausgegeben von F. Klein, in 5 Banden. Leipzig, Teubner, 1909–1916. Band I: Die höheren Schulen in Norddeutschland. Geh. M. 16.00, geb. M. 18.00. Band II: Die höheren Schulen in Süd- und Mitteldeutschland. Geh. M. 12.00, geb. M. 14.00. Band III in 3 Teilbänden: Einzelfragen des höheren mathematischen Unterrichts. 1ter Teil, geh. M. 10.00, geb. M. 12.00; 2ter Teil, 1te Abt., geh. M. 10.00, geb. M. 12.00; 2te Abt., geh. M. 12.00, geb. M. 14.00. Band IV in 2 Teilbanden: Die Mathematik an den technischen Schulen. 1ter Teil, geh. M. 14.00, geb. M. 16.00; 2ter Teil, geh. M. 17.00, geb. M. 19.00. Band V in 2 Teilbanden: Der Mathematische Elementarunterricht und die Mathematik an den Lehrerbildungsanstalten. 1ter Teil, geh. M. 13.00, geb. M. 15.00; 2ter Teil, geh. M. 9.00, geb. M. 11.00.

Carey (F. S.). Infinitesimal calculus. Section 1. London, Longmans, 1917. 8vo. 14 + 144 + 5 pp. 6s.

Eredi (F.). Défaut de tout fondement à la méthode des variations d'Euler et Lagrange: la fonction de maximum et de minimum dans les isopérimètres. Ravenna, impr. Nationale, E. Lavagna et fils, 1917. 8vo. 40 pp.

Filiasi (G.). Appunti di fisica e matematica. (Intorno ad alcune odierne vedute sulla costituzione della materia e sulla interpretazione dei fenomeni naturali; fisica, chimica e metachimica; l'unità delle sensazioni; le matematiche transcendentali e la conoscenza delle cose; gli spazî metageometrici e l'evoluzione degli esseri viventi; la natura e la divinità.) Parte I: Le intuizioni. Vol. 2. Napoli, tip. F. Giannini e figli, 1917. 8vo. 56 + 136 pp. L. 3.00

Gutzmer (A.). Die Tätigkeit des deutschen Unterausschusses. (Berichte und Mitteilungen, veranlasst durch die Internationale Mathematische Unterrichtskommission, herausgegeben von W. Lietzmann, 1te Folge, XII.) Leipzig, Teubner, 1916. 8vo. 31 pp. M. 1.00

Klein (F.). See Abhandlungen.

Lazzeri (G.). Analisi infinitesimale. II: Calcolo integrale. (Biblioteca degli studenti, Volumi 363–365.) 16mo. 6 + 152 pp. L. 1.50

Lietzmann (W.). See Gutzmer (A.).

Morgan (F. M.). See Young (J. W.).

Nelson (A. L.). Plane nets with equal invariants. (Diss., Chicago.) Chicago, University of Chicago, 1916. 8vo. 25 pp.

Rey Pastor (J.). Introducción a la matemática superior. Estado actual, metódos y problemas. Madrid, Biblioteca Corona, 1916. 8vo. 202 pp. Pes. 3.50

——. Teoria de la representación conforme. Conferencias donadas el Juny de 1915, redactadas per E. Terradas. (Publicaciones del 'Instituto de Ciencies. Collecciôn de cursos de fisica i matemàtica dirigida per E. Terradas.) Barcelona, Instituto de Estudios Catalanes, Palacio de la Diputacio, 1917. 8vo. 115 pp. Pes. 3.00

SMITH (D. M.). Jacobi's condition for the problem of Lagrange in the calculus of variations. (Diss., Chicago.) Chicago, University of Chicago, 1916. 4to. 18 pp.

TERRADAS (E.). See REY PASTOR (J.).

WHITEHEAD (A. N.). The organization of thought: educational and scientific. London, Williams and Norgate, 1917. 7 + 228 pp. 6s.

YOUNG (J. W.) and MORGAN (F. M.). Elementary mathematical analysis. New York, Macmillan, 1917. 8vo. 12 + 548 pp. $2.60

II. ELEMENTARY MATHEMATICS.

ATWELL (R. K.). First book in general mathematics. New York, P. P. Simmons Company, 1917. 12mo. 12 + 257 pp. $1.25

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ON INTEGRALS RELATED TO AND EXTENSIONS OF THE LEBESGUE INTEGRALS.

BY PROFESSOR T. H. HILDEBRANDT.

(Read at the Chicago Symposium of the American Mathematical Society, April 6, 1917.)

IN the theory of definite integration, the central position is occupied by the definitions of integration due to Riemann and to Lebesgue. There have however been given a number of other definitions, some of which are related to and in part equivalent to the Lebesgue integral, and some of which are extensions of it; viz., in the direction of giving an integral for a function for which the Lebesgue definition is not applicable either on account of the fact that the set of points at which it is defined is not measurable, or on account of the fact that the function is not absolutely integrable; and in the direction of integration with respect to functions of bounded variation. We purpose in this paper to discuss briefly some of the definitions of integration which have been proposed and consider their relations to the Lebesgue integral. We shall divide the work into four sections.

In the first section we discuss the types of definition of integration which are extensions of the Darboux upper and lower integral method of defining a Riemann integral. Essentially the process involves three steps, (*a*) the selection of a simple class of functions and the definition of integration for functions of this class, (*b*) the extension of this class of functions by the addition of functions which are the limits of sequences converging either uniformly or monotonically, the integral of the limit being defined to be the limit of the integrals, (*c*) the further extension of this class by the addition of all functions $f(x)$ for which there exist functions φ and ψ belonging to the classes defined in (*a*) or (*b*), with equal definite integrals, and such that $\varphi \leqq f \leqq \psi$. We find ideas of this kind in the Lebesgue integration, the last of these steps suggesting, in particular, the distinguishing feature between Borel and Lebesgue measurable sets. Step (*b*), when the convergence is uniform, is present in the Lebesgue definition, while, when we take rather the monotonic sequence, we have the ideas underlying the Young definition which we consider here. This definition is, as we shall show, equivalent to that of Lebesgue in the sense that every function integrable according to either definition is integrable according to the other and the values of the integrals are the same. Very much of the same order is the definition due to Pierpont, which differs from the definition of Young only in that it gives an integral for functions defined on non-measurable sets—assuming that such sets exist. We point out that the Pierpont integral is an extension only in a limited way, in that the functions which are integrable according to this definition are non-measurable in the sense that they are defined on non-measurable sets; in particular we show that if f is Pierpont integrable on a non-measurable set E then there exists a measurable set M containing E and a function φ defined on M which is equal to f on E, such that the Lebesgue integral of φ on M is the same as the Pierpont integral of f on E. Perhaps by stretching the meaning of the word equivalence a little, we might say that the Pierpont and Lebesgue integrals are equivalent.

The second section is devoted to a consideration of the definitions of the integrals of functions which are not integrable according to Lebesgue's definition, because the latter are always absolutely integrable. We consider two types of definitions. The first type depends upon the ability to find a set of intervals whose total length may be made to approach zero, and such that it is possible to define an integral for the function when defined only on the complementary set of points. If these latter integrals approach a limit as the total length of the complementary intervals approaches zero, the limit is the integral over the interval. The integral is thus defined as a result of a single limiting process, and all the singularities are treated simultaneously. The definitions of this type are the Jordan-Harnack-Moore definitions, in which the enclosing intervals are finite in number, and the Borel definitions, in which the intervals may be denumerably infinite in number, i. e., the first are based on content and the second on measure. The other type of definition depends upon the arrangement of the singularities of the function to be integrated in some order, the arrangement and the singularities being dependent upon the integration process used, the ultimate value of the integral being obtained by at most a denumerable infinity of steps and limiting processes. In this category belong the Dirichlet extension of the Cauchy integral for unbounded functions, and the extension of this to the case when the points in the vicinity of which $f(x)$ is not bounded form a reducible set. It is this idea also which Denjoy has applied successfully in order to obtain a definition of integral of unbounded functions which includes as special cases practically all those which have been previously suggested, an integral which proves of importance in the treatment of the relation between a continuous function and its derivative.

The third section is devoted to the Stieltjes integral, which has recently come into the foreground on account of the rôle which it plays in the theory of linear functional operations on continuous functions. We point out that a Stieltjes integral is expressible in terms of a Lebesgue integral of another function, and conversely, but that in spite of this, the Stieltjes integral seems to be applicable where the Lebesgue is not. We give an extension of the Stieltjes integral modelled on the Lebesgue extension of the Riemann integral, as well as the Fréchet generalization of the Lebesgue and Stieltjes integral, so as to apply to a class of general elements. This latter integral depends for its definition on the existence of an absolutely additive function whose range is a set of subclasses of the fundamental class.

The last section gives the definition of the Hellinger integral and also the generalizations of this due to Radon and E. H. Moore.

In the course of the paper we have occasion to point out certain equivalences. They are of two types. The first might be called a complete equivalence, i. e., one in which two definitions of integration yield the same class of integrable functions, and give the same value when applied to a given function. Of this character are the equivalence between the Young and Lebesgue, and one of the Borel definitions and that of Lebesgue;—and we might call attention in passing to what seems to be an indication of the felicity of the Lebesgue definition, that it can be approached from so many distinct and interesting points of view. The other type of equivalence is that which arises in connection with the Stieltjes and Hellinger integrals, each of which is by a transformation reducible to a Lebesgue integral, and conversely; but in either case the function to be integrated is no longer the same. We are inclined to consider this type of equivalence in the light of a pseudo-equivalence, in so far as it is rather a reduction from one form to the other for purposes of evaluation.

The extension of the Lebesgue integral idea in the future will undoubtedly be in the direction of the formulation of a satisfactory integral operation for the case in which the field of integration is a space of a denumerable infinity of dimensions, and the function space. Thus far, there have been given two suggestions for such operations. One is due to Fréchet, who forms, for a general class, the natural extension of the Lebesgue and Stieltjes definition of integration. It depends upon the existence, in a general class of elements, of an absolutely additive function v whose range is a class of subclasses of the fundamental class, i. e., a function such that

$$v(\Sigma_n E_n) = \Sigma_n v(E_n),$$

the E_n being mutually distinct and finite or denumerably infinite in number. The examples of this which have been given for the general space are trivial in that they reduce either to an infinite sum or an integral extended over a field in a finite number of dimensions. There is still lacking a really effective and desirable absolutely additive function for the higher type of spaces. The other suggestion for generalization is due to E. H. Moore. It is essentially an operator of the bilinear or quadratic type, the Fréchet generalization being of the linear type. It depends upon the existence of a function $\epsilon(p, q)$ of two variables p, q, each of which ranges independently over the fundamental class of general elements, and satisfies the condition that for any finite set of elements $p_1, \cdots, p_n$, the quantities $\epsilon(p_i, p_j)$, $i, j = 1, \cdots, n$, form the coefficients of a positively definite Hermitian form in n variables. The operator is determined as a double least upper bound, and is theoretically given as soon as the proper type of ϵ is specified. Moore has given instances of this ϵ in various general spaces which are not trivial. While these two generalizations point the way in which one may go towards the generalization of the Lebesgue integral, they are not entirely satisfactory, the Fréchet generalization, as indicated above, because his instances are trivial, the Moore because it reduces to a Lebesgue integral only after a transformation. There is still room for considerable improvement and investigation in this field.

We restrict ourselves throughout this paper to functions which are single-valued and take finite values at points belonging in most cases to a finite interval (a, b), or a set of points on such an interval. The functions are not, however, necessarily bounded on the interval (a, b). Many of the results are immediately extensible to fields of integration in n-dimensional space. Further, some of the theorems are still valid if $f(x)$ be definitely infinite or many-valued at a set of points of measure zero.

For the sake of convenience we have collected here the most important references which we have consulted in the compilation of this paper, and shall cite them only by number.

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I. The Young and Pierpont Definitions of Integration.

1. *The Darboux Integrals.*—In so far as the Young and Pierpont definitions of integration which we consider in this chapter are extensions of the Darboux treatment of the Riemann integral by means of upper and lower integrals, we recall briefly the definition of these integrals for functions $f(x)$ bounded on an interval (a, b) and a few of their properties.

Definition.—Suppose a partition of the interval (a, b) has been effected by means of the points $a = x_0 < x_1 < \cdots < x_n = b$.

Let M_i be the least upper bound and m_i the greatest lower bound of the values of $f(x)$ on (x_{i-1}, x_i) and form the sums

$$S = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \qquad \text{and} \qquad s = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Then the greatest lower bound of S and the least upper bound of s exist, if we consider all possible partitions of the interval (a, b) into a finite number of subintervals. These bounds are the upper and lower Darboux integrals, respectively, of $f(x)$ on (a, b) and are denoted by

$$\overline{\int_a^b} f(x)dx \qquad \text{and} \qquad \underline{\int_a^b} f(x)dx.$$

We note the following properties:

(1) If $m \leqq f(x) \leqq M$ for x on (a, b), then

$$m(b - a) \leqq s \leqq \underline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x)dx \leqq S \leqq M(b - a).$$

(2) The value of S is not increased, and that of s is not diminished by a repartition of a partition of (a, b).

(3) There exists a sequence of partitions of (a, b) each a repartition of the preceding and with norms approaching zero such that, if s_n and S_n are the corresponding sums s and S, we have

$$\lim_n s_n = \underline{\int_a^b} f(x)dx \qquad \text{and} \qquad \lim_n S_n = \overline{\int_a^b} f(x)dx.$$

Evidently $s_1 \leqq s_2 \leqq \cdots$ and $S_1 \geqq S_2 \geqq \cdots$. Moreover if $\varphi_n(x) = M_{n,i}$ and $\psi_n(x) = m_{n,i}$ in the ith interval $(x_{i-1} \leqq x < x_i)$ of the nth partition, then $\lim_n \varphi_n = \varphi$ and $\lim_n \psi_n = \psi$ exist and we have

$$\psi \leqq f \leqq \varphi; \qquad \text{and} \qquad \int \varphi = \overline{\int} f \qquad \text{and} \qquad \int \psi = \underline{\int} f.$$

(4) A necessary and sufficient condition that $f(x)$ be Riemann integrable is that $\underline{\int} = \overline{\int} = \int$.

From this, other necessary and sufficient conditions for Riemann integrability can be deduced. Perhaps the simplest is the following, due to Lebesgue ((13), page 29):

(5) A necessary and sufficient condition that $f(x)$ be Riemann integrable is that the points of discontinuity of $f(x)$ form a set of measure zero.

2. *The Young Definition of Integration.*—(Cf. Young (26), (27), (28); Nalli (16), pages 97–103.) The Riemann-Darboux definitions of integration depend upon a division of the interval of integration into subintervals; the Lebesgue definition on a division of the interval of variation of the function, which for the interval of integration results in a partition into measurable sets. The question naturally arises whether it is possible to find a definition of Lebesgue integral by replacing in the Darboux definition intervals by measurable sets.* Young ((26), page 243) therefore suggests the following

DEFINITION. *Divide the interval* (a, b) *into a finite or a denumerably infinite number of measurable sets* E_i *of measure* d_i. *Let* M_i *be the least upper bound and* m_i *the greatest lower bound of* $f(x)$ *on* E_i *and form the sums*

$$S = \Sigma_i M_i d_i \qquad \text{and} \qquad s = \Sigma_i m_i d_i.$$

Then the greatest lower bound of S *and the least upper bound of* s *for all possible divisions of* (a, b) *into measurable sets are defined to be the Young (or* (Y)*) upper and lower integrals of* $f(x)$ *on* (a, b). $f(x)$ *is said to be Young (or* (Y)*) integrable if the upper and lower integrals are finite and equal, i. e.,*

$$(Y)\int = (Y)\underline{\int} = (Y)\overline{\int}.$$

Obviously this definition is also applicable in case the interval (a, b) is replaced by any measurable set of points E.

This definition was originally suggested only for functions $f(x)$ bounded on (a, b). In that case the upper and lower integrals always exist, and we obtain the same value for each of them, whether the partition be into a finite or a denumerably infinite number of measurable sets. The same definition will apply also if $f(x)$† is not bounded, provided we assume that $f(x)$ is such that there exist partitions of (a, b) into a denumerable infinity of measurable sets on each of which $f(x)$

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* De la Vallée Poussin (24), pp. 54–56, has pointed out that the Riemann integral is characterized by being a function of intervals, the Lebesgue integral being a function of measurable sets.

† While not expressed explicitly in Young's work, it seems that he considered this extension. Cf., for instance, (28), p. 35.

has a finite upper and lower bound. In determining the upper and lower integrals in this case, we restrict ourselves to partitions of this latter kind.

We are then able to show that, if the upper integral has not the value $-\infty$, the terms of the sum $\Sigma_i M_i d_i$, in which M_i is negative, form a convergent series, i. e., for every such partition, $\Sigma_i M_i d_i$ either diverges to $+\infty$ or converges absolutely. For, if the negative terms of this sum are divergent, then by suitable repartitions of (a, b) and rearrangement of the terms, we can make the sum approach $-\infty$. An analogous result holds for the lower integral, i. e., if it is not $+\infty$, then it either diverges to $-\infty$ or converges. It follows from this that a necessary and sufficient condition that the upper and lower integrals of $f(x)$ be finite is that there exist a partition of (a, b) such that if we form the sum $\Sigma_i M_i d_i$ for $|f(x)|$ this sum is convergent.[*]

In the following list of properties we exclude the case in which $(Y)\overline{\int} = -\infty$ or $(Y)\underline{\int} = +\infty$. Then the proofs are very much as in the case of the Darboux integrals, in so far as the series which enter will be absolutely divergent or convergent. We have

$$(1)\qquad (Y)\overline{\int} \geqq (Y)\underline{\int}.$$

If $f(x)$ is bounded, i. e., $m \leqq f \leqq M$, for every x on (a, b), then

$$m(b-a) \leqq (D)\underline{\int} \leqq (Y)\underline{\int} \leqq (Y)\overline{\int} \leqq (D)\overline{\int} \leqq M(b-a).$$

(2) The values of S are not increased, nor those of s diminished by a repartition of a partition of (a, b) into measurable sets.

(3) There exists a sequence of partitions of (a, b) into measurable sets, each a repartition of the preceding, such that if S_n and s_n are the corresponding values of S and s, then

$$\lim_n S_n = (Y)\overline{\int} \qquad \text{and} \qquad \lim_n s_n = (Y)\underline{\int}$$

and

$$S_1 \geqq S_2 \geqq \cdots \qquad \text{and} \qquad s_1 \leqq s_2 \leqq \cdots.$$

If we let $\varphi_n(x) = M_{n,i}$ and $\psi_n(x) = m_{n,i}$ on the ith set of the nth partition, then the functions $\varphi_n(x)$ will form a monotonic non-increasing sequence and the $\psi_n(x)$ a monotonic non-decreasing sequence of measurable functions such that

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[*] Cf. Fréchet (6), p. 257.

$$\lim_n \psi_n(x) = \psi(x) \leqq f(x) \leqq \varphi(x) = \lim_n \varphi_n(x).$$

Evidently the Lebesgue integrals of $\varphi_n(x)$ and $\psi_n(x)$ exist and are equal to the values of S_n and s_n respectively. Hence by the properties of Lebesgue (or (L)) integrals relative to monotonic sequences (cf. (13), page 98), we have

$$\lim_n S_n = \lim_n (L) \int \varphi_n = (L) \int \varphi = (Y) \overline{\int} f$$

and

$$\lim_n s_n = \lim_n (L) \int \psi_n = (L) \int \psi = (Y) \underline{\int} f.$$

3. *The Relation between the Young and Lebesgue Integrals.*—(Cf. Young (26), pages 243 ff.; Nalli (16), pages 97–106.) A direct consequence of (1) of § 2 is that if $f(x)$ is Riemann integrable, it is also Young integrable. More generally we have:

The Young and Lebesgue definitions of integration are equivalent, and the values obtained by the two definitions the same.

(a) Suppose $f(x)$ (L) integrable on (a, b). Let l_i $(i = -\infty$ to $+\infty)$ be points of division of the interval $(-\infty, +\infty)$. Then the sum

$$\sigma = \Sigma_i l_i \times \text{meas}\,(E(l_i \leqq f < l_{i+1}))$$

will be an s and the sum

$$\Sigma = \Sigma_i l_i \times \text{meas}\,(E(l_{i-1} < f \leqq l_i))$$

will be an S. Hence

$$\sigma \leqq (Y) \underline{\int} \leqq (Y) \overline{\int} \leqq \Sigma.$$

Since σ and Σ approach the $(L)\int$ as the maximum value of the difference $l_{i+1} - l_i$ approaches zero, it is apparent that

$$(L) \int = (Y) \underline{\int} = (Y) \overline{\int} = (Y) \int.$$

(b) On the other hand suppose that $f(x)$ is (Y) integrable on the interval (a, b). Then by property (3) of § 2

$$(L)\int\psi = (Y)\underline{\int} f = (Y)\int f = (Y)\overline{\int} f = (L)\int\varphi,$$

i. e., $(L)\int(\varphi - \psi) = 0$. Hence φ differs from ψ at most at a set of points of measure zero, and consequently, since $\psi \leqq f \leqq \varphi$ at every point, f differs from the measurable and (L) integrable functions φ and ψ at most at a set of measure zero, and hence is measurable and also (L) integrable.

Closely allied with the idea used in this last proof is the following definition of Lebesgue integration due to Riesz ((20), page 453).

The integral is defined first of all for functions which are constant on measurable sets of points. If $f(x)$ is equal to a_i on the set E_i of measure d_i, then

$$\int_a^b f(x)dx = \Sigma_i a_i d_i,$$

the series on the right being assumed to be absolutely convergent if infinite. If we extend this class of functions by the addition of all limits of uniformly convergent sequences of such functions, we get the class of Lebesgue integrable functions. The integral of the limiting function is of course, by definition, the limit of the integrals. This definition lends itself admirably to the proof of theorems of the nature of the Schwarz inequality for summable functions.

In reality the above proof suggests the extension of the class of functions constant on measurable subsets of (a, b) by the addition of the limits of monotonic sequences of these functions, together with the functions which lie between a limit φ of a non-decreasing sequence, and a limit ψ of a non-increasing sequence for which the integrals are equal in value. Young (cf. (27) and (28)) has developed the theory of Lebesgue integration on the basis of monotonic sequences as just sketched, excepting that the initial functions for which integration is defined are upper and lower semi-continuous.

4. *Excursus on Upper and Lower Measure.*—(Cf. Hausdorff (10), pages 408–411.) Before taking up the discussion of the Pierpont integral it may be advantageous to derive a few elementary properties relating to upper and lower measure. We recall that the upper measure of a set is the greatest lower bound of the total lengths of the sets of open intervals which are such that each interval contains at least one point

of the set; that the lower measure is the measure of an interval containing the set, diminished by the upper measure of the complementary set with respect to the interval.

If for two sets E_1 and E_2 we denote by $E_1 + E_2$ their least common superset, and by E_1E_2 their greatest common subset, i. e., their logical sum and product, respectively, we have the following fundamental formula on intervals I_1, I_2:

$$(1)\quad \text{meas}\,(I_1 + I_2) + \text{meas}\,I_1I_2 = \text{meas}\,I_1 + \text{meas}\,I_2,$$

which is immediately extensible to a finite or a denumerably infinite set of intervals. An immediate consequence of this formula is

$$(2)\quad \overline{\text{meas}}\,(E_1 + E_2) + \overline{\text{meas}}\,E_1E_2 \leqq \overline{\text{meas}}\,E_1 + \overline{\text{meas}}\,E_2$$

and by taking complements with respect to an interval including both E_1 and E_2

$$(3)\quad \underline{\text{meas}}\,(E_1' + E_2') + \underline{\text{meas}}\,E_1'E_2' \geqq \underline{\text{meas}}\,E_1' + \underline{\text{meas}}\,E_2',$$

where $E_1' = CE_1$ and $E_2' = CE_2$; i. e., this formula holds for any E_1 and E_2. From these two inequalities we conclude at once that if E_1 and E_2 are measurable, then the sets $E_1 + E_2$ and E_1E_2 are also measurable and

$$(4)\quad \text{meas}\,(E_1 + E_2) + \text{meas}\,E_1E_2 = \text{meas}\,E_1 + \text{meas}\,E_2.$$

From (2) we conclude for every E_1 and E_2

$$(5)\quad \overline{\text{meas}}\,(E_1 + E_2) \leqq \overline{\text{meas}}\,E_1 + \overline{\text{meas}}\,E_2;$$

and from (3), if E_1 and E_2 have no points in common,

$$(6)\quad \underline{\text{meas}}\,(E_1 + E_2) \geqq \underline{\text{meas}}\,E_1 + \underline{\text{meas}}\,E_2.$$

Further, since for every E_1 and E_2

$$CE_1 \leqq C(E_1 + E_2) + E_2,$$

we obtain by applying (5)

$$\overline{\text{meas}}\,CE_1 \leqq \overline{\text{meas}}\,E_2 + \overline{\text{meas}}\,C(E_1 + E_2),$$

or

$$(7)\quad \underline{\text{meas}}\,E_1 + \overline{\text{meas}}\,E_2 \geqq \underline{\text{meas}}\,(E_1 + E_2).$$

Similarly if E_1 and E_2 are distinct, we can apply (6) to

$$CE_1 = C(E_1 + E_2) + E_2$$

and obtain

$$(8)\quad \overline{\text{meas}}\, E_1 + \underline{\text{meas}}\, E_2 \leqq \overline{\text{meas}}\, (E_1 + E_2).$$

If, then, E_1 and E_2 are distinct, we can write down the following series of inequalities:

$$(9)\quad \underline{\text{meas}}\, E_1 + \underline{\text{meas}}\, E_2 \leqq \underline{\text{meas}}\, (E_1 + E_2) \leqq \underline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2 \leqq \overline{\text{meas}}\, (E_1 + E_2) \leqq \overline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2.$$

5. *Relative Measurability; Separated Sets.*—(Cf. Hausdorff (10), pages 415–6; Pierpont (18), pages 366–370.) We define:

E_0 will be said to be *measurable relative to* E, if there exists a measurable set M such that $E_0 = ME$, i. e., E_0 is, so to speak, a section of the set E by a measurable set.

Evidently E is measurable relative to itself. Also, if E_0 is measurable relative to E, then $E - E_0$ will also be measurable relative to E. Finally by (4) of § 4, we have that *if E is measurable, then relative measurability reduces to ordinary measurability.*

We note the following propositions:

(1) If E is divided into a finite or a denumerable infinity of sets E_i measurable relative to E, then

$$\overline{\text{meas}}\, E = \Sigma_i\, \overline{\text{meas}}\, E_i \qquad \text{and} \qquad \underline{\text{meas}}\, E = \Sigma_i\, \underline{\text{meas}}\, E_i.$$

For the case of two sets E_1 and E_2 with $E_1 + E_2 = E$ and $E_1 = ME$, where M is a measurable set, we have by (2) of § 4

$$\overline{\text{meas}}\, M + \overline{\text{meas}}\, E \geqq \overline{\text{meas}}\, (M + E) + \overline{\text{meas}}\, ME = \overline{\text{meas}}\, (M + E_2) + \overline{\text{meas}}\, E_1,$$

and so by (8) of § 4

$$\geqq \underline{\text{meas}}\, M + \overline{\text{meas}}\, E_2 + \overline{\text{meas}}\, E_1.$$

Since M is measurable, we obtain by using (5) of § 4

$$\overline{\text{meas}}\, E = \overline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2.$$

The property for lower measure can be obtained in an analogous way by using (3), (7), and (6) of § 4. The corresponding properties for any finite number, or denumerable infinitude of sets, follow at once from a recurrent use of the result for the case of two sets, and the properties

$$\overline{\text{meas}}\,\Sigma E_i \leqq \Sigma_i\,\overline{\text{meas}}\,E_i;$$

and if $\lim_n E_n = 0$, then $\lim_n \underline{\text{meas}}\,E_n = 0$.

The following converse of this proposition holds:

(2) If E is divided into a finite or denumerable set of sets E_i for which we have $\overline{\text{meas}}\,E = \Sigma_i\,\overline{\text{meas}}\,E_i$, then the E_i are measurable relative to E and $\underline{\text{meas}}\,E = \Sigma_i\,\underline{\text{meas}}\,E_i$.

We indicate the proof for two sets only, the case for a denumerably infinite number of sets being easily deducible from this. Suppose $E = E_1 + E_2$, E_1 and E_2 being distinct, and suppose M_1 and M_2 are measurable sets such that E_1 is contained in M_1 and E_2 in M_2, and meas $M_1 = \overline{\text{meas}}\,E_1$ and meas $M_2 = \overline{\text{meas}}\,E_2$. Then by (4) of § 4

$$\overline{\text{meas}}\,E = \text{meas}\,M_1 + \text{meas}\,M_2 = \text{meas}\,(M_1 + M_2)$$
$$+ \text{meas}\,M_1M_2 \geqq \overline{\text{meas}}\,E + \text{meas}\,M_1M_2,$$

i. e., meas $M_1M_2 = 0$. Hence $\overline{\text{meas}}\,M_2E_1 = 0$, i. e., M_2E_1 is measurable. Consequently, since

$$(M_2 - M_2E_1)E = M_2E - M_2E_1 = E_2,$$

E_2 will be the set common to E and the measurable set $M_2 - M_2E_1$ and so measurable relative to E.

As a consequence of these two propositions we have

(3) If $E_1 \leqq E_2 \leqq \cdots$ are a sequence of sets, each containing the preceding, and each measurable relative to the succeeding set in the sequence, then every set as well as the difference of any two sets, will be measurable relative to $E = \lim_n E_n$.

(4) If M is a measurable set containing E such that meas $M = \overline{\text{meas}}\,E$, then for any division of E into sets measurable relative to E, there exists a division of M into measurable sets M_i such that

$$\text{meas}\,M_i = \overline{\text{meas}}\,E_i \qquad \text{and} \qquad E_i = M_iE.$$

Pierpont ((18), page 366) uses in his work the notion of *separated division*, defined as follows:

E is divided into the separated sets E_1 and E_2 if it is possible to enclose E_1 and E_2 in measurable sets M_1 and M_2 in such a way that meas $M_1M_2 = 0$.*

Pierpont ((18), page 379) shows that if E_1 and E_2 are separated then

$$\overline{\text{meas}}\, E = \overline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2.$$

By using proposition (2) and its proof, we then have:

A necessary and sufficient condition that E_1 and E_2 constitute a separated division of E is that E_1 and E_2 be measurable relative to E, or that

$$\overline{\text{meas}}\, E_1 + \overline{\text{meas}}\, E_2 = \overline{\text{meas}}\, (E_1 + E_2) = \overline{\text{meas}}\, E.$$

Consequently, we have also

$$\underline{\text{meas}}\, E_1 + \underline{\text{meas}}\, E_2 = \underline{\text{meas}}\, E.$$

The same result holds if the division is into a finite or denumerably infinite number of separated sets.

Evidently, then, *if E is measurable, any separated division of E is always into measurable sets.*†

We shall use the term relatively measurable in preference to separated, on account of the close relationship to measurable sets.

6. *The Pierpont Definition of Integration.*—(Cf. Pierpont (18), pages 371 ff.) We are now in a position to give the Pierpont extension of the Young definition of integration.

Suppose E is any set of points on a finite interval, and $f(x)$ defined on E. Divide E into a finite or denumerable set of sets E_i measurable relative to E, and let M_i be the least upper bound, and m_i the greatest lower bound of $f(x)$ on E_i. Form the sums $S = \Sigma_i M_i\, \overline{\text{meas}}\, E_i$ and $s = \Sigma_i m_i\, \overline{\text{meas}}\, E_i$. The greatest lower bound of S and the least upper bound of s for all possible divisions of E into sets measurable relative to E are the upper and lower integrals, respectively, of $f(x)$ on E. $f(x)$ is said to be (P) integrable on E if these two integrals are finite and equal, i. e.,

$$(P)\overline{\int_E} = (P)\underline{\int_E} = (P)\int_E.$$

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* We shall assume that E_1 and E_2 are distinct. Pierpont allows E_1 and E_2 to overlap, but since meas $M_1M_2 = 0$ we have also meas $E_1E_2 = 0$, i. e., they overlap only to the extent of a set of measure zero. The results obtained are the same with either definition, the treatment being slightly simpler as here given.

† It seems that recognition of this fact would have obviated the recent Fréchet-Pierpont controversy in these columns, vol. 22, 295–302; vol. 23, 172–5. Cf. also W. A. Wilson, vol. 22, pp. 384–386.

Evidently when the set E is measurable this definition is identical with that of Young. Although stated only for bounded functions by Pierpont, it is immediately applicable to functions not bounded on E. As a matter of fact the remarks made in connection with the Young integral can be immediately transferred to the Pierpont integral if we replace measurable by relatively measurable, and measure by upper measure. In particular, if we say that $f(x)$ is *measurable relative to* E (Pierpont, separable, cf. (18), page 403), if the set E_l of points of E for which $f(x) > l$ is measurable relative to E, we can build up an integral along the lines of Lebesgue. If we call a function integrable in this way *summable relative to* E, we have the theorem:

(1) *Pierpont integrability on* E *is equivalent to summability relative to* E, *the values obtained for the two integrals being the same.**

We note further the property

(2) If E is divided into a set of sets E_i measurable relative to E, and if f is integrable on each E_i, then f is also integrable on E and $\int_E = \Sigma_i \int_{E_i}$.

On the other hand we have the following:

(3) If f is (P) integrable on E, and $E_0 < E$, then f is also (P) integrable on E_0.

This holds, if we restrict ourselves to (Y) integrals, only in case both E_0 and E are measurable.

On account of the fact that, for a division of the set E into sets measurable relative to E, we also have

$$\underline{\text{meas}}\, E = \Sigma_i\, \underline{\text{meas}}\, E_i,$$

we can obtain a definition of integration on sets E by replacing, in the Pierpont integral, upper measure by lower measure. If we call integrability after the manner of this last definition (P_0) integrability, then if $f(x)$ is (P) integrable on E it is also (P_0) integrable. The converse does not seem to hold, because a set of lower measure zero does not necessarily have zero measure.

While the Pierpont integral gives an integral for a function defined on a non-measurable set, it is an extension of the

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* Cf. Lamond, *Transactions Amer. Math. Society*, vol. 16 (1915), p. 393, where we find a similar theorem for f bounded.

Lebesgue definition only in a limited sense. For if we enclose any non-measurable set E in a measurable set M of such a nature that meas $M = \overline{\text{meas}}\, E$, then there exists on M a function $f_0(x)$ measurable on M and equal to $f(x)$ on E, such that

$$(L) \int_M f_0(x) = (P) \int_E f(x);$$

i. e., a function $f(x)$ measurable relative to a set E is a section of a measurable function defined on a measurable set including E. By using (4) of § 5, we can construct the function $f_0(x)$ by taking the function φ of § 2, corresponding to f, but defined on M, and modifying it at the points of E of measure zero, in such a way that it agrees with f on E.

We gain insight into the Pierpont integral also from the geometrical point of view. If $f > 0$, and we consider the planar set $[x \text{ on } E,\ 0 \leqq y < f(x)]$, then the (P) integral gives us the upper planar measure of this set, the (P_0) integral the lower planar measure. These will be equal for $f > 0$ only if the set E for which $f > 0$ is measurable, i. e., f is Lebesgue integrable.

The (P) integral has proved of value in connection with the reduction of a double integral to an iterated integral, in so far as a linear section of a planar measurable set is not necessarily linearly measurable.

It seems, however, that the question of giving a definition for the integration of essentially non-measurable functions, i. e., for instance functions defined on a measurable set which are not measurable, has not yet been satisfactorily solved.

7. *The Pierpont Definition of Integration for Unbounded Functions.*—(Cf. (8), pages 405 ff.) We note finally Pierpont's treatment of integrals of functions which are not bounded on a set E, very briefly. Let $E_{\alpha\beta}$ $(\alpha, \beta > 0)$ be the subset of E for which $-\beta \leqq f(x) \leqq \alpha$. Suppose that the $(P) \int_{E_{\alpha\beta}} f(x)dx$ exists for every $\alpha, \beta > 0$. Then $\int_E$ is defined to be the

$$\lim_{\alpha,\, \beta \to \infty} \int_{E_{\alpha\beta}}$$

provided this limit exists and has a finite value.

This definition is equivalent to the one given above in § 6. For, if we assume that $(P)\int_E f$ exists, then f will be relatively measurable on E, and hence $(P)\int_{E_{\alpha\beta}}$ will exist for every α and β. We show that

$$\lim_{\alpha,\,\beta \doteq \infty} (P)\int_{E_{\alpha\beta}} = (P)\int_E$$

by using the property (2) of § 6. On the other hand if $(P)\int_{E_{\alpha\beta}}$ exists for every α, β, then we can show that the $E_{\alpha\beta}$ are measurable relative to E, and to each other by using property (3) of § 5, from which we conclude that f is measurable relative to E. The existence of

$$(P)\int_E = \lim_{\alpha,\,\beta} \int_{E_{\alpha\beta}}$$

follows from property (2) of § 6.

II. The Borel and Denjoy Definitions of Integration.

Suppose that, in an interval (a, b), $f(x)$ is not bounded in every vicinity of a point c; suppose further that the definite (Riemann) integral exists in the intervals $(a, c - \epsilon')$ and $(c + \epsilon'', b)$ for every ϵ' and ϵ''. Then consistent with the continuity of the definite integral as a function of its limits we define (as suggested by Cauchy)

$$\int_a^b f(x) = \lim_{\epsilon' \doteq 0} \int_a^{c-\epsilon'} f(x) + \lim_{\epsilon'' \doteq 0} \int_{c+\epsilon''}^b f(x).$$

Evidently the extension of this to the case in which the number of points of (a, b) in every vicinity of which $f(x)$ is not bounded is finite, is immediate.

When the number of points in the vicinity of which $f(x)$ is not bounded becomes infinite then there are two types of definition. One of them gives a definition by means of a single limiting process, the other by a denumerable set of such processes. The first of these leads to the Harnack-Jordan-Moore and Borel types of integration, the other to the Dirichlet, extended by Hoelder and Lebesgue, and Denjoy definitions of integration.

1. *The Harnack-Jordan Definition.*—(Cf. Harnack (9), pages 220 ff.; Jordan (12), pages 50, 51; Moore (15); Nalli (16), pages 20–27.) We note first of all that the set of points in every vicinity of which $f(x)$ is not bounded constitute a closed set. We call this set for the present the set of singularities, and denote it by Z. Then Harnack defines the integral of $f(x)$ as follows:

Suppose the set Z of singularities of $f(x)$ is of content zero. Enclose them in a finite set of intervals of total length ϵ. Let $f_1(x)$ be zero in the interior of the enclosing intervals, and equal to $f(x)$ everywhere else and suppose that $\int_a^b f_1(x)dx$ exists. If this integral approaches a finite limit as ϵ approaches zero, this limit is said to be the integral of $f(x)$ from a to b.

Jordan gives a definition which is equivalent to this in case the content of Z is zero. His definition is as follows:

Divide (a, b) into any finite number of intervals of maximum length δ. Exclude the intervals containing points of the set Z, and suppose that the (Riemann) integrals of $f(x)$ exist on the remaining intervals. If the sum of these integrals approaches a definite finite limit when δ approaches zero, this is defined to be the integral of $f(x)$ from a to b.

Moore ((15), pages 300–302) observes that the Harnack definition can be applied when the set Z is replaced by another Z_0 containing it, and that the resulting integral is in reality a function of the set Z_0. Further, that in case the set of singularities Z is non-existent then the integral of $f(x)$ on the basis of the set Z_0 is equal to the ordinary integral of $f(x)$ if and only if the set Z_0 is of content zero. For that reason it is desirable to restrict the consideration of these integrals to sets Z of content zero, and it is to be supposed that Jordan assumed this to be the case even though he does not state this fact explicitly.*

2. *The Borel Integral.*—(Cf. Borel (2*a*), pages 199–205; (2*b*), pages 249–252; Hahn (8).) Borel has given a definition similar to that of Harnack except that he assumes that the set of singularities Z is of measure zero, and may be everywhere dense, i. e., in general not closed. In this case Z is not necessarily the set of all points in every neighborhood of which the function $f(x)$ becomes unbounded.

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* No change, however, is made in the definition in the 3d edition of his Cours d'Analyse

Before taking up the definition which Borel gives, it may be desirable to observe that it is possible to define a Riemann integral of a function $f(x)$ on a perfect or closed set P. Viz., if in the sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

we replace the length of the interval (x_i, x_{i-1}) by the content h_i of the part of P contained in the interval, and choose ξ_i to belong to the ith interval and P, then the limit of this sum as the length of the maximum subinterval approaches zero is the Riemann integral of $f(x)$ on P. We note that a necessary and sufficient condition for the existence of a Riemann integral on P is that the set of points of P at which $f(x)$ is discontinuous of measure be zero.

Borel's definition is then equivalent to the following

DEFINITION. *Suppose the set Z of singularities is of measure zero, and such that if it is enclosed in a set of intervals of total length ϵ, each interval containing at least one point of the set Z, then the Riemann integral of $f(x)$ exists on the complementary set P_ϵ. If*

$$\lim_{\epsilon=0} \int_{P_\epsilon} f(x)$$

exists, for all possible enclosures of the type described, it is defined to be the (B) integral of $f(x)$ on (a, b).

Borel assumes that this definition is to be applied to functions which are not bounded on (a, b), but it can obviously be applied also to functions which are bounded. The relation of this definition to the one of Lebesgue has been discussed by Hahn (8). He proves the following theorems:

(1) If $f(x)$ is absolutely (B) integrable, i. e., if $(B)\int |f(x)|\,dx$ exists, then $f(x)$ is also Lebesgue integrable.

Apparently this includes as a special case that in which $f(x)$ is a bounded function.

By way of proof we note that $(R)\int_{P_\epsilon} = (L)\int_{P_\epsilon}$ for any f for which the first integral exists. Let now $\epsilon_1, \cdots, \epsilon_n, \cdots$, be such that $\lim_n \epsilon_n = 0$, and let $P_n = P_{\epsilon_n}$ be the corresponding closed sets of measure greater than $b - a - \epsilon_n$, which are chosen in such a way that P_n contains P_{n-1}. If then $f_n = f$ on P_n and zero everywhere else, then the $|f_n|$ will form a monotonic non-decreasing sequence of functions approaching $|f|$ except possibly at a set of measure zero. Hence by the properties of (L) integrals

$$(B)\int_a^b |f| = \lim_n \int_{P_n} |f| = \lim_n\, (L)\int_{P_n} |f|$$
$$= \lim_n\, (L)\int_a^b |f_n| = (L)\int_a^b |f|.$$

From this we conclude the existence of $(L)\int_a^b f$ and since meas P_n approaches $b - a$, its equality to $(B)\int_a^b f$.

(2) A necessary and sufficient condition under the hypothesis of the definition that f be (B) integrable is that (a) f be (L) integrable, i. e., summable, on the set $Z + Z'$, where Z' is the derived set of Z; and (b), if $u_n = (a_n, b_n)$ are the intervals complementary to the closed set $Z + Z'$, and $W(u_n)$ is the maximum value of $\left|\int_{u_n'} f\right|$ for all intervals u_n' interior to u_n, then $\Sigma_n W(u_n)$ shall form a convergent series. Moreover

$$(B)\int_a^b f(x)dx = \Sigma_n \int_{u_n} f(x)dx + (L)\int_{Z+Z'} f(x)dx,$$

where

$$\int_{u_n} = \lim_{u_n' \to u_n} \int_{u_n'}.$$

We refer to the original memoir of Hahn for the proof of this theorem. In case the set $Z + Z'$ is of measure zero, and therefore of content zero, the Borel definition may be reduced to that of Harnack-Moore and the above equality lacks the second term on the right, in which form it was originally derived by Moore (cf. (15), pages 324 ff.). We shall refer to this theorem as the Moore-Hahn theorem.

(3) *The Borel definition is not equivalent to the Lebesgue definition of integration, in particular there exist functions which are Borel integrable without being Lebesgue integrable, and functions which are Lebesgue integrable without being Borel integrable.*

For, as given, the definition of Borel includes functions which are not absolutely integrable (cf. Moore (15), page 327), while any Lebesgue integrable function is absolutely integrable. On the other hand the function $f(x)$, which is zero except at the points of a non-dense perfect set of measure greater than zero, at which it has the value 1, though (L) integrable, is not (B) integrable. For Hahn has shown that there does not exist any set of measure zero such that for every enclosure of this set, each interval of the enclosure containing a point of this set, the function $f(x)$ is Riemann integrable on the set of points complementary to the intervals.

We can give a definition which will include as special cases the Lebesgue and Borel integrals by assuming that $f(x)$ is bounded and Lebesgue integrable on the sets P_ϵ and that the integral on the interval is the limit of the integrals on the sets P_ϵ. Then a theorem similar to the Moore-Hahn theorem will hold, the integrals $\int_{u_n'}$ being now taken in the sense of Lebesgue.

A further extension is to assume that on the sets P_ϵ, $f(x)$ is not necessarily bounded, but only Lebesgue summable. That this actually is an extension can be shown by a simple example.

Suppose that the interval is (0, 1) and $f(x)$ is defined as follows: (a) in the interval $i_n = \left(\frac{1}{n}, \frac{1}{n+1}\right)$, $f(x) > 0$ if n is even and $f(x) < 0$ if n is odd;

(b) $$\left|\int_{i_n} f(x)dx\right| = \frac{1}{n};$$

(c) $f(x)$ is continuous throughout i_n except that

$$\lim_{\epsilon=0}\left|f\left(\frac{1}{n}+\epsilon\right)\right| = \infty.$$

Then it can be shown* that the Borel integral of this function in the interval (0, 1) does not exist, the set of singularities being the points $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\cdots$, 0; if, however, we apply the last definition suggested above with the set of singularities $Z = (0)$, we obtain an integral.

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* Cf., for instance, Moore (15), p. 329.

We note finally that the Harnack-Borel integrals do not have the property which we find in the case of Lebesgue and Riemann integrals that if f and g are integrable, then $f + g$ is also integrable. Borel has suggested that in such a case we set by definition

$$\int (f+g) = \int f + \int g.$$

3. *The* (B^*) *Integral.*—(Cf. Hahn (8), page 9; Nalli (16), pages 84–97.) In the Borel definition of integral we note (a) that the set of singularities Z is a fixed set of points (i. e., in a way the integral is a function of the set Z); (b) every allowable set of intervals is such that each interval of the set contains at least one point of the set Z. A definition has been suggested* in which both of these requirements are dropped. It is as follows:

DEFINITION. *Suppose* (a) $f(x)$ *is such that for every* $\epsilon > 0$, *there exists a perfect set* P_ϵ *of measure greater than* $b - a - \epsilon$, *for which the Riemann integral* $\int_{P_\epsilon} f$ *exists; further suppose* (b) *that*

$$\lim_{\epsilon \doteq 0} \int_{P_\epsilon} \overline{\overline{f}}$$

exists. Then this limit is said to be the (B^*) *integral of* $f(x)$ *on* (a, b).

Obviously since for every ϵ we can find in any measurable set E a perfect set P_ϵ such that meas E − meas $P_\epsilon < \epsilon$ we can apply the same type of definition to the definition of an integral extended over a measurable set E.

We note the following propositions:

(1) A necessary and sufficient condition that there exist a (B^*) integral is that for every $\epsilon > 0$ there exists a δ_ϵ such that if P_1 and P_2 have measures greater than $b - a - \delta_\epsilon$, and $\int_{P_1}$ and $\int_{P_2}$ exist, then

$$\left| \int_{P_1} - \int_{P_2} \right| \leqq \epsilon.$$

(2) If the (B^*) integral of $f(x)$ exists on (a, b), then for any sequence of ϵ's such that $\epsilon_1 \geqq \epsilon_2 \geqq \cdots$ and $\lim_n \epsilon_n = 0$

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* Cf. Hahn (8), p. 9. Nalli (16), p. 85, erroneously asserts that Borel's definition is the one which we now give.

we can find a sequence of perfect sets P_n of measure greater than $b - a - \epsilon_n$, each containing the preceding, such that $\int_{P_n}$ exists and $\lim_n \int_{P_n} = (B^*) \int_a^b$.

For this purpose we obtain a series of P_ϵ: $P_{\epsilon_1 - \epsilon_2}$, $P_{\epsilon_2 - \epsilon_3}$, $\cdots$. For P_n we take the greatest perfect set common to the sets $P_{\epsilon_m - \epsilon_{m+1}}$, for $m \geqq n$. This will evidently have measure greater than $b - a - \epsilon_n$, and be contained in P_{n+1}.

(3) If $f(x)$ is bounded in (a, b) and fulfils condition (a) of the definition, then the (B^*) integral exists.

(4) If f and g are (B^*) integrable, then $f + g$ is (B^*) integrable and we have

$$(B^*) \int (f + g) = (B^*) \int f + (B^*) \int g,$$

a proposition which does not hold for the (B) integral.

(5) *If f is integrable according to the Lebesgue definition, then f is also (B^*) integrable, the values of the integrals being the same.*

For if f is (L) integrable, then f is measurable. From the theorem of Lusin* on measurable functions:

If f is measurable in an interval (a, b), then for every $\epsilon > 0$ there exists a non-dense perfect set P_ϵ of measure greater than $b - a - \epsilon$ on which f is continuous,

it follows that the condition (a) of the definition is satisfied, i. e., for every $\epsilon > 0$ there exists a P_ϵ of measure greater than $b - a - \epsilon$ for which the $(R) \int_{P_\epsilon}$ exists. Evidently

$$(R) \int_{P_\epsilon} = (L) \int_{P_\epsilon}.$$

Moreover we know that the (L) integral is absolutely continuous, i. e., if $\lim_n \text{meas } E_n = \text{meas } E$, then $\lim_n (L) \int_{E_n} = (L) \int_E$. Hence since

$$\lim_{\epsilon = 0} \text{meas } P_\epsilon = b - a,$$

it follows that

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* Cf. *Comptes Rendus*, vol. 154 (1912), p. 1689. Cf. also Lebesgue, *Comptes Rendus*, vol. 137 (1903), p. 1229.

$$\lim_{\epsilon \doteq 0} (R) \int_{P_\epsilon} = \lim_{\epsilon \doteq 0} (L) \int_{P_\epsilon} = (L) \int_a^b ,$$

i. e., the (B^*) integral exists and is equal to the (L) integral. The same method of reasoning would show that if $(L) \int_E$ exists, where E is any measurable set, then $(B^*) \int_E$ exists also.

We note in passing that the Hahn example of a function not (B) integrable is obviously (B^*) integrable.

On the other hand we also have

(6) *If* $f(x)$ *is* (B^*) *integrable on* (a, b), *then it is also* (L) *integrable.*

We observe that if we show that an f which is (B^*) integrable is absolutely (B^*) integrable, i. e., $(B^*) \int |f|$ exists, we can apply the method of proof used for (1) of § 2 above, for the case of (B) integrability. We therefore proceed to show

(7) If $f(x)$ is (B^*) integrable then $|f|$ is also (B^*) integrable.

Suppose if possible that $|f|$ is not (B^*) integrable. Then from (2) above it would follow that $(B^*) \int |f|$ is infinite. Hence if $f_1 = \frac{1}{2}(|f| + f)$ and $f_2 = \frac{1}{2}(|f| - f)$, then by (4) $(B^*) \int f_1$ and $(B^*) \int f_2$ are both infinite. For every M we can then determine a perfect set $P_{1\epsilon}$ of measure greater than $b - a - \epsilon$ such that $\int_{P_{1\epsilon}} f_1 > M$. For the same ϵ we can determine a perfect set $P_{2\epsilon}$ on which $(R) \int_{P_{2\epsilon}} f_2$ exists, and consequently f_2 is bounded on $P_{2\epsilon}$. Suppose $f_2 < m$ on $P_{2\epsilon}$. Then $(R) \int_{P_{2\epsilon}} f_2 < m(b - a)$. Let the set E be the sum of the measurable sets: (a) the subset of $P_{1\epsilon}$ for which $f_1 > 0$ and hence $f_2 = 0$; (b) the subset of $P_{2\epsilon}$ for which $f_2 > 0$; (c) the greatest common subset of $P_{1\epsilon}$ and $P_{2\epsilon}$. Then the measure of E will be greater than $b - a - 2\epsilon$ and

$$(L) \int_E f = (R) \int_{P_{1\epsilon}} f_1 - (R) \int_{P_{2\epsilon}} f_2 > M - m(b - a).$$

However, by (5) we can find a perfect set P_ϵ of measure greater than or equal to $b - a - 2\epsilon$, contained in E, such that the difference between $(L)\int_E f$ and $(R)\int_{P_\epsilon} f$ is as small as we please, i. e., such that

$$(R)\int_{P_\epsilon} f \geqq M - m(b - a).$$

Since M has been chosen arbitrarily, it follows that

$$\lim_{\epsilon \doteq 0} \int_{P_\epsilon} f$$

does not exist, i. e., f is not (B^*) integrable, which is a contradiction. Hence the theorem.

We thus have the result

(8) *(B^*) integrability is equivalent to (L) integrability, the values of the integrals being the same.*

The absolute convergence of the (B^*) integral is another instance of a phenomenon observed by Stolz[*] and Moore.[†] In the (B^*) integration while the singularities are enclosed in a set of intervals of total length less than ϵ, it is not required that every interval contain a point of the set, i. e., we have a *broad* enclosure. On the other hand, in the (B) integration, every interval must include a point of the set of singularities, i. e., we have a *narrow* enclosure. The broad enclosures yield absolutely convergent integrals, the narrow enclosures may lead to non-absolutely convergent integrals.

4. *The Dirichlet-Lebesgue Extension of the Cauchy Definition.*—(Cf. Lebesgue (13), pages 9–14.) The Cauchy definition of integral of a function $f(x)$, which has only a finite number of points in every neighborhood of which $f(x)$ is not bounded, can be extended so as to give the Harnack and Borel definitions of integration as a result of a single limiting process. We turn now to the other type of definition which requires a finite or denumerable infinitude of limiting processes.

The first step in this direction was taken by Dirichlet, who extends the Cauchy definition to the case in which the interval (a, b) has a set of singularities Z whose derived set Z' contains a finite number of points $x_1, \cdots, x_n$. For in any subinterval interior to the interval $(x_{\nu-1}, x_\nu)$ we have only a finite number of singularities. To find $\int_{x_{i-1}-\epsilon}^{x_i+\epsilon'} f(x)dx$, for any two positive ϵ and ϵ', we can then apply the Cauchy definition. If the limit of this integral exists as ϵ and ϵ' approach zero, it is the integral in the interval (x_{i-1}, x_i), and the integral in (a, b) is the sum of these integrals.

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[*] *Wiener Berichte*, vol. 108_{2a} (1899), p. 1235.

[†] (5), pp. 302–304, 323.

We observe that the Cauchy-Dirichlet definitional process makes use of the following two conditions:

(*a*) If an integral exists on each of a finite set of consecutive intervals (a_1, a_2); (a_2, a_3); $\cdots$; (a_{n-1}, a_n), then

$$\int_{a_1}^{a_n} = \sum_{i=1}^{n-1} \int_{a_i}^{a_{i+1}}.$$

(*b*) If $\int_{a+\epsilon'}^{b-\epsilon''}$ exists for every ϵ' and ϵ'', and if this integral approaches a limit as ϵ' and ϵ'' approach zero, this limit is defined to be $\int_a^b$.

Evidently the Dirichlet manner of reasoning is immediately extensible to the case in which Z'' and then $Z^{(\alpha)}$, α being any finite or transfinite number of the first and second classes, contains a finite number of points,* i. e., if Z is a closed reducible set, then this method will give a definition of integration over the interval (a, b). Lebesgue† shows that this same result may be attained by the following definition:

A function $f(x)$ has an integral in the finite interval (a, b) if there exists a continuous function $F(x)$, unique except for the addition of a constant, such that we have in every interval (α, β) where $f(x)$ is continuous

$$\int_\alpha^\beta f(x)dx = F(\beta) - F(\alpha).$$

$F(x)$ is the indefinite integral of $f(x)$ and we have

$$\int_a^b f(x)dx = F(b) - F(a).$$

In applying the process just discussed we start from $Z^{(\alpha)}$, and work back to Z by a denumerable set of limiting processes, in every interval into which $Z^{(\alpha)}$ divides (a, b). We can obtain the same result in case Z is reducible, by starting with Z and working up to $Z^{(\alpha)}$. For consider the intervals complementary to Z'. By the two conditions of the Cauchy-Dirichlet process, we can then define an integral in each of these intervals. Evidently the same holds true of the intervals complementary to Z'', and so on. By a well-known theorem in the theory of point sets* we arrive finally at the intervals complementary to a set $Z^{(\alpha)}$ which is either zero or perfect. If it is zero, then we have an integral for the entire interval. If perfect, then we have obtained an integral in each interval complementary to this perfect set $Z^{(\alpha)} = P$. If $W(u_n)$ is the maximum absolute oscillation of $\int_{u_n'}$, u_n' being any interval interior to u_n, and $\Sigma W(u_n)$ is convergent, and P is of content zero, then the Moore-Hahn theorem (2) of § 2 would suggest that we define

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* Cf. Schoenfliess, Bericht über die Mengenlehre, I (1900), p. 185.

† Cf. (13), pp. 10–14. See also Holder, *Math. Annalen*, vol. 24 (1884), pp. 90 ff.

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_{u_n} f(x)dx.$$

We observe that this will include the Harnack integral as a special case.†

We might proceed to a discussion of the case in which the content of P is not zero, or the series of oscillations or integrals over the complementary intervals u_n is not absolutely convergent, but we prefer to take up briefly the Denjoy definition of integration which combines the two definitional conditions of Cauchy-Dirichlet with the Moore-Hahn theorem, and includes as special cases almost all of the definitions so far considered in this chapter.

5. *The Denjoy Definition of Integration.*—(Cf. Denjoy (3); Nalli (16), pages 122 ff.) We give first a few definitions.

(a) A function $f(x)$ is said to be *not summable at a point* x_0 of an interval, if $f(x)$ is not summable in every interval containing the point x_0. Similarly $f(x)$ is not summable at a point x_0 on a perfect set P, if the function $\varphi = f$ on P and zero at points not on P is not summable at x_0. In either case the points at which $f(x)$ is not summable form a closed set.

(b) Let $[u_n]$ be the set of intervals complementary to a perfect set P, relative to an interval (a, b), and suppose that for every n there is defined a constant $W(u_n)$. Then the series $\Sigma W(u_n)$ is said to be *absolutely convergent on an interval* δ if the part of the series $\Sigma W(u_n)$ which corresponds to the intervals interior to δ is absolutely convergent. It is *absolutely convergent at a point* x_0 of the perfect set P if there exists an interval containing x_0 in which $\Sigma_n W(u_n)$ is absolutely convergent. Evidently the points of P at which $\Sigma_n W(u_n)$ is not absolutely convergent form a closed set.

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* Cf. Hobson, Theory of Functions, pp. 92, 93.

† Cf. Moore (15), p. 330.

Then Denjoy defines an integral $V(a, b)$ on the interval (a, b) as follows:

(A) The integral of $f(x)$ in an interval in which $f(x)$ is summable is the Lebesgue integral.

(B) If the integral has been defined for a finite number of consecutive intervals: (a_1, a_2); (a_2, a_3); $\cdots$; (a_{n-1}, a_n) then $V(a_1, a_n)$ is defined to be

$$V(a_1, a_n) = \sum_{i=1}^{n-1} V(a_i, a_{i+1}).$$

(C) If (a) $f(x)$ is summable on a perfect set P, whose complementary intervals relative to an interval (c, d) are u_n; (b) $V(u_n')$ has been defined on every interval u_n' containing no points of P in its interior and $W(u_n)$ is the least upper bound of $|V(u_n')|$ for all possible u_n' interior to u_n; and (c) $\Sigma_n W(u_n)$ is convergent, then

$$V(c, d) = \Sigma_n V(u_n) + \int_P f(x)dx.$$

The function $f(x)$ is said to be *Denjoy* (*or* (Dn)) *integrable*, if the following conditions are fulfilled:

I. The set of points of any perfect set P at which $f(x)$ is not summable on P is not dense on P.

II. If $V(c', d')$ has been defined for every interval (c', d') interior to $V(c, d)$, then

$$\lim_{c \doteq c',\, d \doteq d'} V(c', d')$$

exists and is defined to be $V(c, d)$.

III. The set of points of any perfect set P at which the series of maximum oscillations $W(u_n)$ of V on the intervals complementary to P is not absolutely convergent is not dense on P.

The proof that these three definitional conditions and the three integrability conditions suffice to determine for a function $f(x)$ a finite number, its integral, is rather long. We mention only the chief features of the proof.

A central position is occupied by the theorem on point sets: If we have a series of closed sets $E^{(\alpha)}$, corresponding to the transfinite numbers of the first and second class, each contained in the preceding and not dense on it, then there exists a transfinite number α with a precedent, such that $E^{(\alpha-1)} \neq 0$ and $E^{(\alpha)} = 0$.

Condition I when applied to the interval (a, b) gives a closed set E_1, not dense on (a, b) at every point of which f is not summable on (a, b). We break up E_1 into $P_1 + R_1$, where P_1 is perfect and R_1 is reducible. Then we replace (a, b) by P_1 and apply condition I again. This yields $E_2 = P_2 + R_2$. And so we continue. The theorem just given shows that this process must terminate after a denumerably infinite number of steps.

Condition II, with the definitional condition B, when applied to the intervals complementary to E_1 gives a value for the intervals complementary to the derived set E_1' of E_1. The same two conditions lead us to the intervals complementary to E_1'' and eventually to the intervals complementary to P_1.

In the same way, definitional condition C together with the conditions II and III give us a value for $V(g, h)$ in any interval (g, h) which contains points of P_1 but not of E_2, i. e., in any interval complementary to E_2. Similarly we find the value of $V(g, h)$ for every interval complementary to $E_{\alpha+1}$, the values in the intervals complementary to P_α having been determined.

We thus obtain a value for the integral $V(a, b)$ in the interval (a, b) after a denumerably infinite number of steps.

We note the following properties:

(1) If $f(x)$ is such that $|f(x)|$ is Denjoy integrable, then $f(x)$ is Lebesgue integrable.

For evidently if $f(x) \geqq 0$, and there existed a point at which $f(x)$ were not summable on (a, b), then $V(a, b)$ would also be infinite.

(2) From the Moore-Hahn theorem of § 2 it follows that if $f(x)$ is (B) integrable it is also Denjoy integrable, the definitional condition C being an immediate extension of the expression for the (B) integral. The same result holds true

if in the (B) integral we replace the Riemann integrals by Lebesgue integrals. Just how the definitions compare when Riemann integrability on P_ϵ is replaced by Lebesgue summability, we do not know, except that there are functions which are Denjoy integrable without being (B) integrable in any extensions of the definition, an immediate consequence of

(3) If f and g are Denjoy integrable, then $f + g$ is also Denjoy integrable, and we have

$$(Dn)\int f + (Dn)\int g = (Dn)\int (f + g).$$

We have further

(4) The indefinite Denjoy integral $V(a, x)$ is a continuous function and has as derivative $f(x)$ except for a set of measure zero.

(5) If the continuous function $F(x)$ has at every point a finite derivative $f(x)$ then this derivative is Denjoy integrable, and the Denjoy integral of $f(x)$ has the value $F(x) - F(a)$.

This result is of importance in the theory of the derivatives of continuous functions. In the case in which the derivative is finite at every point, it gives a means of returning from the derivative to the original function. For the Riemann integral, the derived function $f(x)$ must be Riemann integrable; for the Lebesgue integral, $f(x)$ must be summable, and this carries with it that $F(x)$ be of bounded variation; but $f(x)$, finite at every point, is Denjoy integrable.*

For the proofs of these theorems we refer to the Denjoy articles, as well as the Nalli thesis, which contains a careful analysis of the Denjoy first notes on the subject.

Quite recently Khintchine† has pointed out that in order to obtain an integral by the Denjoy method of definition, it is sufficient to replace the condition of convergence of the maximum oscillations $W(u_n)$ in the intervals u_n, in the definitional condition C and the integrability condition III, by the weaker condition of convergence of the series of variations of the integrals $\Sigma_n V(u_n)$, in as much as $\Sigma_n V(u_n)$ and not $\Sigma_n W(u_n)$

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* Cf. in this connection Lusin: *Comptes Rendus*, volume 155 (1912), pp. 1475–7. See also *Annali di Matematica*, ser. 3, vol. 26 (1917), pp. 77–131. Pp. 118–131 contain a very interesting suggestion for extending the definition of integration by the use of Fourier series.

† "Sur une extension de l'intégrale de M. Denjoy," *Comptes Rendus*, vol. 162 (1916), pp. 287–291. Cf. also pp. 377–380.

appears in the integration expression. However, for an integral defined in this way it is no longer true that the derivative of the indefinite integral $V(a, x)$ is the function $f(x)$ except for a set of measure zero. By defining the term "integrals approaching zero asymptotically at a point x_0 of a perfect set P" to mean that if d_n is the distance between x_0 and the nearest extremity of the interval u_n complementary to P, then

$$\lim_n \frac{W(u_n)}{d_n} = 0,$$

he shows that a necessary and sufficient condition that the derivative of $V(a, x)$ be $f(x)$ except for a set of measure zero, is that the definitional condition C be replaced by

C'. If f is summable on P in an interval (c, d), if, moreover, the integrals approach zero asymptotically except for a set of measure zero on P, then

$$V(c, d) = (L) \int_P f + \Sigma_n V(u_n);$$

and the integrability condition III by

III′. For every perfect set P, there exists a portion* π of P such that if $\delta_1, \cdots, \delta_n, \cdots$ are the intervals complementary to π, the series $\Sigma_n V(\delta_n)$ converges and the integrals approach zero asymptotically on π except for a set of measure zero on π.

Both of the Khintchine suggestions give actual extensions of the Denjoy definition, i. e., give an integral for a function not integrable by the definition as given by Denjoy.

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* Cf., for instance, Denjoy (4(a)), p. 120.

(*To be continued.*)

A CONTINUOUS FUNCTION WHOSE DEVELOPMENT IN BESSEL'S FUNCTIONS IS NON-SUMMABLE OF CERTAIN ORDERS.

BY PROFESSOR CHARLES N. MOORE.

(Read before the American Mathematical Society, September 4, 1916.)

THE first example of a continuous function whose Fourier's development diverges at one or more points was given, as is well known, by Du Bois-Reymond.[*] Much more recently Haar[†] showed how to construct a continuous function whose Sturm-Liouville development is divergent, and a continuous function whose development in Legendre's functions has this same property. Somewhat later Gronwall[‡] gave in explicit form a function whose development in Legendre's functions is not summable (Ck) for $0 \leqq k \leqq \frac{1}{2}$, at the point $x = 1$.

As far as the writer is aware, no examples have thus far been given of continuous functions whose developments in Bessel's functions diverge or fail to be summable of certain orders at one or more points. In the present paper there is exhibited in explicit form a continuous function whose development in Bessel's functions of order zero is not summable (Ck) for $0 \leqq k < \frac{1}{2}$, at the point $x = 0$. This function is analogous to a function given by Fejér[§] whose Fourier development diverges for $x = 0$. The proof of the non-summability, however, follows different lines from Fejér's proof of the divergence for his example.

If we represent by A_r the rth coefficient in the development of an arbitrary function $f(x)$ in terms of Bessel's functions, we have

$$A_r = \frac{\int_0^1 xf(x)J_0(\lambda_r x)dx}{\int_0^1 xJ_0^2(\lambda_r x)dx}, \tag{1}$$

where λ_r is the rth root of the equation

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[*] *Abhandlungen der Bayerischen Akademie der Wissenschaften*, vol. 12 (1876).

[†] Inaugural Dissertation, Göttingen (1909); also *Math. Annalen*, vol. 69 (1910).

[‡] *Math. Annalen*, vol. 75 (1914).

[§] *Journal für die reine und angewandte Mathematik*, vol. 137 (1910).

$$l\lambda J_0'(\lambda) + hJ_0(\lambda) = 0, \tag{2}$$

l and h being any constants not both zero. Moreover, we have for λ_r

$$\lambda_r = (r + q)\pi + \frac{\psi(r)}{r}, \tag{3}$$

where q is a constant and $\psi(r)$ is used to represent a function of r that remains finite for all values of r.

Consider now the function $f(x)$ defined by the following equation

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{x} \cos (2^n + q)\pi x}{n^2}, \tag{4}$$

where q is the q of (3). We will show that for the coefficients A_r in the development of this function in terms of Bessel's functions

$$\lim_{m \to \infty} \frac{A_{2^m}}{(2^m)^k} = \infty \qquad (0 \leqq k < \tfrac{1}{2}). \tag{5}$$

Thus it will follow that the series

$$A_1 + A_2 + A_3 + \cdots + A_r + \cdots \tag{6}$$

is not summable (Ck) for $0 \leqq k < \frac{1}{2}$, since for any series so summable the limit on the left-hand side of (5) exists and is equal to zero.* But since $J_0(0) = 1$, the development of $f(x)$ in Bessel's functions of order zero reduces to the form (6) for $x = 0$, and therefore it will have been shown that this development is not summable (Ck) for $0 \leqq k < \frac{1}{2}$ at the point $x = 0$

It only remains then to establish the relationship (5). Since the series on the right-hand side of (4) is uniformly convergent in the interval $0 \leqq x \leqq 1$, we have from (1)

$$A_r = \frac{1}{\displaystyle\int_0^1 xJ_0^2(\lambda_r x)dx} \sum_{n=1}^{\infty} \frac{\displaystyle\int_0^1 x^{\frac{3}{2}} \cos (2^n + q)\pi x J_0(\lambda_r x)dx}{n^2}. \tag{7}$$

From the asymptotic expansion of $J_0(x)$ we have

$$J_0(\lambda x) = \sqrt{\frac{2}{\pi\lambda x}} \cos\left(\lambda x - \frac{\pi}{4}\right) + \frac{\psi(\lambda x)}{(\lambda x)^{\frac{3}{2}}}. \tag{8}$$

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* Cf. Chapman, *Proc. London Math. Society*, ser. 2, vol. 9 (1911), p. 379.

Making use of (8), we see that the general term of the series on the right-hand side of (7) may be written in the form

$$
(9)\qquad \begin{aligned} &\frac{1}{n^2}\sqrt{\frac{2}{\pi\lambda_r}}\int_0^1 x\cos\left(\lambda_r x-\frac{\pi}{4}\right)\cos\,(2^n+q)\pi x dx \\ &\qquad\qquad +\frac{1}{n^2\lambda_r^{3/2}}\int_0^1 \cos\,(2^n+q)\pi x\psi(\lambda_r x)dx. \end{aligned}
$$

If now we set $r = 2^m$ and make use of the relation (3), it is readily seen that the sum of the series whose general term is the second term of (9) is of the form

$$
(10)\qquad O\left(\frac{1}{\lambda_r^{3/2}}\right) = O\left(\frac{1}{\sqrt{\lambda_r}}\cdot\frac{1}{2^m}\right) = o\left(\frac{1}{\sqrt{\lambda_r}}\cdot\frac{1}{m^2}\right),
$$

where we have used $O(x)$ to represent any function which after division by x remains finite as x becomes infinite, and $o(x)$ to represent any function which after division by x approaches zero as x becomes infinite.*

We have further to consider the series whose general term is the first term of (9). Making use of (3) we may write

$$
(11)\qquad \begin{aligned} &\cos\left(\lambda_r x-\frac{\pi}{4}\right) \\ &= \cos\frac{\pi}{4}\cos\,(r+q)\pi x+\sin\frac{\pi}{4}\sin\,(r+q)\pi x \\ &\quad +\cos\left\{(r+q)\pi x-\frac{\pi}{4}+\frac{\psi(r)}{r}x\right\}-\cos\left\{(r+q)\pi x-\frac{\pi}{4}\right\}, \end{aligned}
$$

and the last two terms on the right-hand side of (11) may be written in the form

$$
(12)\qquad -\frac{\psi(r)}{r}x\sin\left\{(r+q)\pi x-\frac{\pi}{4}+\theta\right\}\quad\left(|\theta|<\left|\frac{\psi(r)}{r}x\right|\right).
$$

Substituting from (11) in the first term of (9) and making use of (12), we reduce this term to the form

$$
\frac{1}{n^2\sqrt{\pi\lambda_r}}\int_0^1 x\cos\,(2^m+q)\pi x\cos\,(2^n+q)\pi x dx
$$

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* The use of O in this sense is due to Bachmann. Cf. Analytische Zahlentheorie, vol. 2, p. 401. The corresponding use of o is due to Landau. Cf. Handbuch der Lehre von der Verteilung der Primzahlen, vol. 1, p. 61.

$$
(13) \qquad \begin{aligned}
&+\frac{1}{n^2\sqrt{\pi\lambda_r}}\cdot\int_0^1 x \sin\,(2^m+q)\pi x \cos\,(2^n+q)\pi x dx \\
&-\frac{\psi(2^m)}{2^m\cdot n^2}\cdot\sqrt{\frac{2}{\pi\lambda_r}}\int_0^1 x^2 \sin\left\{(2^m+q)\pi x\right. \\
&\qquad\qquad\qquad\qquad \left.-\frac{\pi}{4}+\theta\right\}\cos\,(2^n x+q)\pi x dx.
\end{aligned}
$$

The last term in (13) is obviously the general term of a series whose sum is of the form (10). Consider next the first term in (13). If $n \neq m$, the integral in this term reduces to the form

$$
\frac{\cos\,(2^m+2^n+2q)\pi-1}{2(2^m+2^n+2q)^2\pi^2}+\frac{\sin\,(2^m+2^n+2q)\pi}{2(2^m+2^n+2q)\pi}
$$
$$
+\frac{\cos\,(2^m-2^n)\pi-1}{2(2^m-2^n)^2\pi^2}+\frac{\sin\,(2^m-2^n)\pi}{2(2^m-2^n)\pi}.
$$

This quantity is obviously less in absolute value than $K/|2^m-2^n|$, where K is a positive constant. Moreover, for all values of $n < m$

$$2^m-2^n > 2^n(2^{m-n}-1) \geqq 2^n\cdot 2^{m-n-1} = 2^{m-1},$$

and for all values of $n > m$

$$2^n-2^m \geqq 2\cdot 2^m-2^m = 2^m.$$

Hence it follows that $K/|2^m-2^n| < K/2^{m-1}$, and therefore the sum of all the terms of the form of the first term in (13) for which $n \neq m$ is of the form $O(1/(\sqrt{\lambda_r}\cdot 2^{m-1}))$ and consequently of the form of the last member of (10).

If $n = m$, the integral in the first term of (13) reduces to

$$
\frac{\cos 2(2^m+q)\pi-1}{8(2^m+q)^2\pi^2}+\frac{\sin 2(2^m+q)\pi}{4(2^m+q)\pi}+\frac{1}{4},
$$

and hence for this value of n the first term of (13) takes the form

$$
(14) \qquad o\left(\frac{1}{\sqrt{\lambda_r}}\cdot\frac{1}{m^2}\right)+\frac{1}{4\sqrt{\pi}\sqrt{\lambda_r}m^2}.
$$

Combining this with the previous result, we see that the series of which the first term of (13) is the general term has a sum which is of the form (14).

We have finally to discuss the second term of (13). The integral in this term may for $n \neq m$ be reduced to the form

$$\frac{\sin(2^m+2^n+2q)\pi}{2(2^m+2^n+2q)^2\pi^2}-\frac{\cos(2^m+2^n+2q)\pi}{2(2^m+2^n+2q)\pi}$$
$$+\frac{\sin(2^m-2^n)\pi}{2(2^m-2^n)^2\pi^2}-\frac{\cos(2^m-2^n)\pi}{2(2^m-2^n)}.$$

For $n = m$ it takes the form

$$\frac{\sin 2(2^m+q)\pi}{8(2^m+q)^2\pi^2}-\frac{\cos 2(2^m+q)\pi}{4(2^m+q)\pi}.$$

Hence the series whose general term is the second term of (13) has a sum which is of the form of the last member of (10).

We have now seen that the series whose general terms are the second and third terms of (13), respectively, have sums that are of the form of the last member of (10), whereas the series whose general term is the first term of (13) has a sum that is of the form (14). Hence the series whose general term is (13), or the first term of (9), has a sum which is of the form (14). Since, moreover, the series whose general term is the second term of (9) has a sum which is of the form (10), it follows that the series whose general term is (9), that is, the series in the right-hand member of (7), has a sum which is of the form (14). As the other factor of the right-hand member of (7) is of the form $C\lambda_r + o(\lambda_r)$, where C is a positive constant, it follows that

$$(15)\quad A_r = C_1\frac{\sqrt{\lambda_r}}{m^2}+o\left(\frac{\sqrt{\lambda_r}}{m^2}\right) = C_2\frac{\sqrt{r}}{m^2}+o\left(\frac{\sqrt{r}}{m^2}\right)$$
$$= C_2\frac{\sqrt{2^m}}{m^2}+o\left(\frac{\sqrt{2^m}}{m^2}\right),$$

where C_1 and C_2 are positive constants and we make use of the fact that we have chosen $r = 2^m$.

It follows from (15) that A_r satisfies the relationship (5), and hence as pointed out before, *the development in Bessel's functions of order zero of the function* $f(x)$ *defined by* (4) *is not summable* (Ck) *for* $0 \leqq k < \frac{1}{2}$ *at the point* $x = 0$.

UNIVERSITY OF CINCINNATI.

FINITE GROUPS.

Theory and Applications of Finite Groups. By G. A. MILLER, H. F. BLICHFELDT, and L. E. DICKSON. New York, Wiley, 1916. 8vo. 17 + 390 pp.

DURING the last half-century the theory of groups has been explored with great vigor and success, and it is now an imposing body of doctrine. Its importance is principally due to the fact that it enters into the very warp and woof of a large number of mathematical textures. "Many branches of mathematics are nothing but the theory of invariants of special groups." This is especially true of geometry, as was made clear once for all by Klein. Geometry is not one science, but an infinite variety of sciences, depending on the choice of the group on which it is based. As the fundamental group becomes larger and more extensive, the corresponding geometry becomes less extensive and more fundamental. Thus, beginning with the elementary geometry of congruent figures, we obtain in succession a number of different geometries among which are the geometry of similar figures, affine geometry, projective geometry, conformal geometry, and analysis situs.

From a purely abstract standpoint groups may be classified in various ways: they are either abelian or non-abelian, finite or infinite, continuous or discontinuous. The principal classes of concrete groups are permutation groups, linear groups, and groups of non-linear transformations, together with numerous geometrical and mechanical applications. Even in the study of point sets and other aggregates groups have been found useful; for instance, in Bernstein's proof that if two cardinal numbers $n\alpha$ and $n\beta$ are equal, then $\alpha = \beta$.

In a linear group the transformations may be homogeneous or fractional, singular or non-singular, but the most striking principle of classification is that which depends on the nature of the coefficients, which may belong to an infinite field, a finite field, or no field whatever. If the coefficients belong to an infinite field or domain, for instance that of all complex numbers, we obtain a most important class of groups, both continuous and discontinuous, with which are closely associated groups of collineations and other geometric transformations. If the coefficients belong to a finite (or Galois) field, of which a special case is that in which they are residues with respect to a prime modulus, the groups are themselves necessarily finite. But the coefficients may not belong to any field. Thus they may be residues with respect to a composite modulus, or with respect to a number of different moduli, or with respect to a modular system. Very little has been done toward the investigation of this class of groups.

When we come to groups of non-linear and in particular of non-algebraic transformations, the possible developments are too vast to contemplate and for the most part lie buried in the future. In the case of birational transformations a beginning has been made. The tremendous variety of geometric groups is evident when we recall that the geometry under consideration may be non-euclidean as well as euclidean, that its number of dimensions may be n or even infinite, that the elements which are transformed may be not merely points, but lines, circles, spheres, etc., and finally that the set of such elements, say of points, instead of forming a continuum, may form a point set of the most general type. Since many physical and chemical changes also form groups, Lie was almost justified in his enthusiastic query, "What are the phenomena of the physical and material universe but the transformations of an enormous group, of which the laws of nature are the invariants?"

Even that part of the theory which has arrived at a fair degree of completeness and finish is far too extensive for a single treatise. The volume before us is restricted to finite groups and indeed to certain selected topics in their theory and applications; linear congruence groups are barely touched upon. The book is written by three recognized authorities on various phases of the subject. Considerable original material is included by each of the authors either in the form of new developments or of simplifications in the proofs of the older theorems. The work is strictly up to date; it contains some of the very latest results on the topics treated.

G. A. Miller, who is responsible for Part I, on abstract and permutation groups, is thoroughly at home in this field; indeed, he has contributed so extensively to its theory and has created so many new and valuable phases of it that he may fairly be called one of the world's leading authorities on the subject. Part II, by Blichfeldt, is devoted to finite groups of linear homogeneous transformations whose coefficients belong to the domain of complex numbers. The writer is eminently fitted for this task, because he is himself responsible for a new and brilliant method of attack, which enabled him to arrive at the first exhaustive enumeration and discussion of the groups in four variables. Part III, by Dickson, covers the Galois theory of equations and applications to certain geometric problems. In this case also the writer is thoroughly qualified for his task. For he is a very productive mathematician in this and various other fields, who has not only given a definitive and apparently final form to the theory of linear congruence groups and thereby added tremendously to our knowledge of simple groups, but has also contributed largely to the progress of mathematics in many other directions.

Naturally the result of this kind of collaboration is a volume consisting practically of three separate treatises, and yet a certain amount of unity has been attained; while Parts II and III are largely independent of one another, they both presuppose a knowledge of some of the more elementary sections of Part I. We already possess treatises and textbooks on finite groups by Jordan and de Séguier in French, by Bianchi in Italian, by Klein, Weber, and Netto in German, and by Burnside, Hilton, and Dickson in English. But this new volume will easily bear comparison with any of them, not only in its wealth of valuable material, much of which is new, but in vigor and directness of approach, in elegance of treatment, and in the thoroughly interesting way in which many of the results are presented. Certain flaws, mostly of a secondary nature, will presently be pointed out, but every reader must admit the general excellence of the work. There is a graceful and becoming dedication to Camille Jordan, " whose fundamental investigations in the theory and applications of finite groups enriched the subject to the extent of converting it into a fundamental branch of mathematics and furnished in a large measure the inspiration for the subsequent great activity in this field."

In Part I, which covers about half the book, the systematic exposition is preceded, in Chapter 1, by a few simple examples of concrete groups, including the groups of rotations of the equilateral triangle and of the square, a trigonometric illustration of the octic group, and a few simple congruence groups and groups of matrices. In this way the wind is very properly tempered to the shorn lamb, namely, the beginner. Another admirable feature, pedagogically, is the fact that substitution (or permutation) groups are studied before abstract groups. Indeed, one has the feeling, in reading Part I, that every abstract group is clothed in the flesh and blood of a substitution group, and in particular of a regular group.

In Chapter 2 we meet with substitution groups and their properties, including transforms, co-sets, Sylow's theorem, transitive and intransitive groups, primitive and imprimitive groups, the simplicity of the alternating group, the class of a group, the holomorph of a group, and a determination of all the groups of degree $\leqq 5$. Chapter 3 is devoted to the more fundamental theorems on abstract groups, including the cyclic, dihedral, and dicyclic groups, commutator subgroups, ϕ-subgroups, inner isomorphisms, and the representation of a group as a transitive or regular substitution group. In Chapter 4 we come to abelian groups, their invariants, characteristic subgroups, and groups of isomorphisms; also the roots of their elements and the number of elements of a given order; abelian groups which are conformal with non-abelian groups, non-abelian groups in which every subgroup is abelian, and Hamilton groups, in which every subgroup is invariant. The bitter pill of pure abstraction is sugar-coated for us with the delectable group of totitives. Chapter 5 is concerned with prime-power groups, their subgroups of various types, their groups of isomorphisms, and finally a method of constructing all the possible groups of order p^m. A proof is given of the important new theorem that the number of elements in a set of independent generators of a prime-power group is an invariant of the group. In Chapters 6 and 7 the results are more largely due to Miller's own research than in the other chapters, and are of extremely great value and interest. Chapter 6 is devoted to groups having simple abstract definitions and includes the groups generated by two elements having a common square, the groups of the regular polyhedrons, and various generalizations of the latter. Chapter 7 contains an intensive study of groups of isomorphisms. Chapter 8 is concerned with series of composition, chief series, solvable groups, and unsolvable groups of low composite orders.

Now for a few adverse criticisms. The author of Part I does not always get in touch with the beginner; some of the initial difficulties of the subject are slurred over. For instance, on page 16, in the third sentence, the adverb "evidently" might easily be regarded as ironical by the beginner. Moreover, the fourth sentence on the same page is not true, unless the cycles are assumed to have no letters in common and therefore to be different from the cycles in the product written down in the next paragraph. A number of terms and phrases are not carefully or explicitly defined; e. g., "transitive constituent groups" on page 33, line 25, "$(1, m)$ correspondence" on page 39, line 1, the "holomorph" and "group of isomorphisms" of an abstract group. The fact that a simple isomorphism is a (1, 1) correspondence in which the products of corresponding elements are themselves corresponding elements is nowhere explicitly mentioned; and this is all the more misleading since the term (1, 1) correspondence is used as synonymous with simple isomorphism. In the middle paragraph of page 103 it is implied that a subgroup can be generated by operators not belonging to the subgroup. On pages 34 and 37 no explanation is given as to what is meant by "regarding several substitutions as one substitution."

The following erroneous or ambiguous statements have been noted. On page 70, lines 9, 10, the elements of the commutator are not to be permuted in every *possible* manner, as there stated, but only in such a way that it may remain a commutator. On page 34, line 3 from the bottom, "the symmetric group" should read "two symmetric groups." On page 53, line 3 from the bottom, "of the form $l + mk$" should read "$l + m_1k$ and $l + m_2k$." In § 38, second paragraph, first sentence, the adjective "low" should be inserted before "order," in order to avoid an apparent conflict with page 89, lines 20–26. On page 99, first theorem, last line, "equal to or greater than" should read "divisible by." On page 179, line 5, "α conjugates of H_1" should read "$\alpha - 1$ terms." On page 185, lines 12, 13, "whose order exceeds unity" should read "which is non-invariant." On page 113, last line, the article "the" should be omitted, as it implies that there is only one Sylow subgroup of each given order. The first sentence on page 174 contains the erroneous statement that the subgroups there mentioned are invariant; but the error is corrected by implication in the next paragraph.

Part II, consisting of Chapters 9–13, gives us a more direct approach to the finite groups of linear transformations, especially those in three variables, than is contained in any earlier text. We start off, in Chapter 9, along a well-beaten track, which takes us to the characteristic equation, Hermitian invariants, unitary transformations, reducible or intransitive groups, the canonical form, etc. In Chapter 10 we encounter the groups in two variables, which are determined by Klein's original geometric method, somewhat simplified. Thus we get a second view of the groups of the regular polyhedrons. Chapters 11 and 12 contain the author's own powerful method of handling the groups in three variables. As compared with the older methods of Jordan, Valentiner, and others, it is remarkable for its brevity and completeness. An important rôle is played by Kronecker's theorem on roots of unity, and also by the theorem that no prime > 7 can divide the order of a primitive group in three variables. In Chapter 13 we have a brief introduction to the theory of group characteristics and one or two of its applications.

The last formula on page 230 would be somewhat clearer if it were stated that α_i is a multiplier of T_i and that the notation $\alpha_i(y_1, y_2)$ means $(\alpha_i y_1, \alpha_i y_2)$. A misleading reference to "regular polygons" is made on page 219, line 14; for of course in the case of the dihedral groups the polygons are not regular. An unusual and rather unfortunate nomenclature is employed in § 77, line 1, and also in Part III, on page 373, line 7. While a set of values of the unknowns satisfying a system of equations, one for each unknown, is commonly called a solution of the system, the authors call it a set of solutions, the value of each unknown being a solution. In the definition on page 194 the condition that the equations "can be solved" for the new variables is hardly drastic enough; what is meant is that the equations must possess exactly one solution (or set of solutions, to use the author's language).

Part III, consisting of Chapters 14–20, begins with the Galois theory of equations, the treatment of which is somewhat similar to that of the author's excellent "Theory of Algebraic Equations," except that it is simpler and briefer, and that the comparatively easy case in which the domain of rationality involves only constants is wisely given first. Chapters 15 and 16, on the necessary and sufficient conditions for solvability by radicals, including a discussion of the cyclotomic equation and of the general cubic and quartic, naturally connect up with the chapter on solvable groups in Part I.

We now come to some interesting and valuable applications which show what a powerful tool the theory of groups is in solving many problems in geometry. In Chapter 17 we have constructions with ruler and compasses, in Chapter 18 the nine inflection points of a plane cubic curve, and in Chapter 19 the twenty-seven straight lines on a cubic surface as determined by means of their double-six configurations and the twenty-eight bitangents to a quartic curve as determined by means of their Steiner sets and Aronhold sets. The intimate connection between the last two problems is well shown. For the bitangents there is given an exhibition of Hesse's and Cayley's unsymmetric notation and also a purely algebraic derivation of the symmetric notation which was originally derived from the properties of theta functions. The methods of proof are largely the author's own and as compared with earlier treatments like that in Weber's Algebra, for instance, they show a vast improvement. Especially noteworthy is the determination of the number of real elements in each case.

The necessary geometric theorems are not presupposed, but are derived in the text; so that the discussion is intelligible to a reader with no knowledge of homogeneous coordinates or projective geometry; or at least it would be intelligible to him if some explanation had been given of the so-called points and lines at infinity, and if the cases in which these *ideal* elements are included had been carefully distinguished from those in which they are not. This has not been done; hence a certain amount of confusion. E. g., in § 176, fourth sentence, we are supposed to pass from a triangle of reference to Cartesian coordinates by setting $x_2/x_1 = x$, $x_3/x_1 = y$. But this special kind of triangle of reference has not been defined in the text. Moreover, in the definition of the ordinary triangle of reference, on page 327, the condition $\Delta \neq 0$ is not sufficient, for it would allow two sides of the triangle to be parallel. Again, on page 346, line 1, the surface may be a hyperbolic paraboloid as well as a hyperboloid of one sheet, even when the lines c_2, c_3, c_4 are all real. In the first paragraph of § 187, in order to avoid infinite values of m or ξ, the x-axis would have to be chosen so as to be neither parallel nor perpendicular to any bitangent.

The following typographical errors have been noticed.

Page 24, second footnote, second line. For "her" read "here."
Page 48, line 26. For "s_1" read "s_2."

Page 68, line 21. For "congredient" read "cogredient."
Page 140, line 9. Interchange "order" and "product."
Page 163, line 24. For "operator" read "operators."
Page 183, line 24. For "subgroups" read "subgroup."
§ 91, line 2. For the second "w" read "$\overline{w}$."
Page 212, exercise 1. After "transformation" insert "of determinant unity."
Page 345, footnote, line 5. For "c_3" read "c_4."
Page 381, line 6 from bottom. For "tenary" read "ternary."

In spite of a few slight blemishes, the book is an extremely valuable contribution to mathematical literature, one that for many years to come will be indispensable to the worker in the domain of finite groups.

ARTHUR RANUM.

SHORTER NOTICES.

L'Annuaire du Bureau des Longitudes pour l'An 1917. Paris, Gauthier-Villars.

AN interesting feature of the *Annuaire* for 1917 is the extensive "Notice"—"L'Avance de l'heure légale pendant l'été de l'année 1916," by M. J. Renaud. In its ninety-odd pages he traces the history of daylight-saving from the initiation of the mean time of Paris as the legal hour for France and Algeria to the passing of the law last year which put the clocks forward one hour during the summer months. A resumé of the opinions of scientific bodies, of the press, of the Chamber of Commerce of Paris, and even of the Académie des Sports on the last change is followed by reports of the discussions which took place in the Senate and Chamber of Deputies; there is much material here for those who are interested in the possible adoption of a similar change on this continent. A further section is devoted to the various measures which were taken to avoid confusion in making the change of hour in the spring and fall, and still another section to what has been done in other countries. M. Renaud concludes finally that concerted measures by international agreement are necessary if proper stability in the indication of legal time is to be effected.

In the other "Notices" M. G. Bigourdan gives an account of the Babylonian calendar in some detail, M. M. Hamy sums up what has been done to determine the length of the meter in terms of wave lengths of light, and M. Renaud writes of the life and work of the hydrographic engineer, Ph. Hatt, who died in October, 1915.

The body of the volume contains a few changes which are mainly minor additions to and improvements in the astronomical portions which fill over half its pages in the odd years. No lessening of the care which has always been bestowed in making the *Annuaire* useful and up to date is apparent.

ERNEST W. BROWN.

Theory of Errors and Least Squares. By LEROY D. WELD. New York, Macmillan, 1916.

PROFESSOR WELD'S volume is mainly intended for those research workers who wish to use the method of least squares in dealing with their observations, but who have little knowledge of mathematics. With this end in view, he has attempted to give the necessary formulas in the simplest possible form and has added numerous examples drawn from problems in chemistry, physics, astronomy, geodesy, statistics, etc. Recognizing, however, that the competent observer is rarely willing to use a formula without knowing the basis on which it is constructed, the author devotes the earlier part to a detailed and simple explanation of the principles on which the theory is constructed.

He has attained considerable success in this effort. Nevertheless there is one point of view on which more explanation seems to be advisable. The theory of errors of observation, based on the least square principle, has become so standardized that those who use it much rarely need to remember that its results and technical terms rest on a set of assumptions which, it is true, are founded on experience, but which have not the definite character of a physical law deduced from observations. The beginner, whether an observer or a student learning the subject, needs to have the fact impressed on his mind that the whole theory with its technical terms such as "probable error of an observation," "most probable value" very largely form a language through which an observer may communicate briefly the results of his work and the degree of consistency of his observations. The theory gives no assurance that physical laws are correctly represented. This is perhaps rather a point of view than a criticism, but it is one which frequently creates difficulties and its absence from most of the published treatments of the theory of errors is partly responsible for the unwillingness of many observers to use the methods of least squares.

A useful feature of the book is the Appendix, which contains some of the mathematical deductions which, placed in the body of the work, would perhaps have repelled or frightened the student. The various technical terms, rules and formulas are also gathered together ready for use.

ERNEST W. BROWN.

CARSLAW'S NON-EUCLIDEAN GEOMETRY.

I SHOULD like to point out that Professor Coolidge has quite misunderstood the definition of "nominal length" to which he refers on page 466 of his review of my little book on Non-Euclidean Geometry in the July BULLETIN; and he has failed to notice the indication I give at the beginning of § 94 of "just how a nominal line *corresponds* to a rectilinear segment."

As a matter of fact the full discussion of the euclidean case was given in my paper in the *Proceedings of the Edinburgh Mathematical Society,* and in Appendix V to the English translation of Bonola's book, both of which are mentioned in the footnote to page 156 of the book under review. It seemed unnecessary to repeat this introductory passage in full. In giving an abstract of it, the process of condensation has obviously been carried too far.

But a glance at one or other of the passages referred to will show that I am not guilty of the "lamentable" error with which your reviewer credits me.

H. S. CARSLAW.

THE UNIVERSITY OF SYDNEY,
August 10, 1917.

NOTES.

THE annual meeting of the American Mathematical Society will be held in New York City on Thursday and Friday, December 27–28. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by December 3, if the abstracts are to be printed in advance of the meeting. Other titles may be sent in not later than December 10.

THE fortieth regular meeting of the Chicago Section will be held at the University of Chicago on Friday and Saturday, December 28–29. Friday afternoon will be devoted to a joint meeting with the Mathematical Association of America, at which Professor W. B. FORD will deliver his retiring address as chairman of the Section. The joint dinner of the two organizations will be held on Thursday evening. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary of the Section by December 3.

THE concluding (October) number of volume 18 of the *Transactions of the American Mathematical Society* contains the following papers: "Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points," by C. E. WILDER; "A memoir on the doctrine of associated forms," by O. E. GLENN; "The degree and class of multiply transitive groups," by W. A. MANNING; "Some geometric characterizations of isothermal nets on a curved surface," by G. M. GREEN; "A new method in boundary problems for differential equations," by R. G. D. RICHARDSON; "Note supplementary to the paper 'On certain pairs of transcendental functions whose roots separate each other,'" by MAXIME BÔCHER; "A theorem for space analogous to Cesàro's theorem for plane isogonal systems," by P. F. SMITH.

THE concluding (October) number of volume 39 of the *American Journal of Mathematics* contains the following papers: "The potential of a lens, and allied physical problems," by

G. GREENHILL; "On the asymptotic character of functions defined by series of the form $\Sigma\, c_n g\, (x + n)$," by R. D. CARMICHAEL; "Possible characteristic operators of a group," by G. A. MILLER; "Linear differential equations in infinitely many variables," by W. L. HART; "On the asymptotic value of sums of power residues," by H. H. MITCHELL; "On the structure of finite continuous groups," by J. A. BULLARD.

THE following mathematical papers have appeared in recent numbers of the *Proceedings of the National Academy of Sciences:* in the April, 1917, number (volume 3, number 4), "Dynamical systems with two degrees of freedom," by G. D. BIRKHOFF; in the July number (volume 3, number 7), "The Cayleyan curve of the quartic," by TERESA COHEN, and "Triads of transformations of conjugate systems of curves," by L. P. EISENHART; in the October number (volume 3, number 10), "On the general theory of curved surfaces and rectilinear congruences," by G. M. GREEN.

MANUSCRIPTS intended for publication in the *Bulletin de la Société mathématique de France* may hereafter be written in French, English, or Italian.

THE library of the American Mathematical Association has been deposited in the library of Oberlin College.

THE following 29 doctorates with mathematics as major subject were conferred by American universities in the academic year 1916–1917; the title of the dissertation is added in each case: C. C. BRAMBLE, Johns Hopkins, "A complete system for a collineation group isomorphic with the group of the double tangents of a plane quartic"; C. C. CAMP, Cornell, "An extension of the Sturm-Liouville expansion"; SARAH E. CRONIN, Columbia, "Geometric properties completely characterizing all the curves in a plane along which the constrained motion is such that the pressure is proportional to the normal component of the acting force"; W. L. CRUM, Yale, "The perturbations caused by a close approach of two asteroids"; MARY F. CURTIS, Harvard, "Curves invariant under point transformations of special type"; T. DANTZIG, Indiana, "Contributions to the theory of plane transformations"; L. R. FORD, Harvard, "On rational approximations to an irrational complex number"; E. D. GRANT, Chicago, "The motion of a flexible cable in a vertical plane"; C. M.

HEBBERT, Illinois, "Projective pencils of stelloids and the curve generated by them"; A. W. HOBBS, Johns Hopkins, "On a problem of projectiles"; T. R. HOLLCROFT, Cornell, "On the types of plane (2, 3) correspondences"; GOLDIE P. HORTON, Texas, "Lebesgue integrals"; M. T. HU, Harvard, "Linear integro-differential equations with a boundary condition"; E. P. HUBBLE, Chicago, "Photographic investigations of faint nebulæ"; K. W. LAMSON, Chicago, "A general implicit function theorem with an application to problems of relative minimum"; GILLIE A. LAREW, Chicago, "Necessary conditions for the problem of Mayer in the calculus of variations"; FLORA E. LESTOURGEON, Chicago, "Minima of functions of lines"; A. S. MERRILL, Chicago, "An isoperimetric problem with variable end points"; B. E. MITCHELL, Columbia, "Complex conics and their real representation"; H. C. M. MORSE, Harvard, "Certain types of geodesic motion on a surface of negative curvature"; G. W. MULLINS, Columbia, "Differential invariants under the inversion group"; I. C, NICHOLS, Michigan, "A comparative study on fractions in the early treatises on the Hindu art of reckoning"; J. F. RITT, Columbia, "On a general class of linear homogeneous differential equations of infinite order with constant coefficients preceded by a note on the resolution into partial fractions of the reciprocal of an entire function of genus zero"; E. F. SIMONDS, Columbia, "Differential invariants in the plane"; T. McN. SIMPSON, Chicago, "Relations between the metric and projective theories of space curves"; D. M. SMITH, Chicago, "Jacobi's condition for problems of Lagrange in the calculus of variations"; G. W. SMITH, Illinois, "On nilpotent algebras generated by two units such that i^2 is not an independent unit"; E. A. WHITE, Cornell, "Mechanical theory of the plow"; W. H. WILSON, Illinois, "On a certain class of functional equations."

RICE INSTITUTE. The following advanced mathematical courses are given during the present academic year.—By Professors P. J. DANIELL and G. C. EVANS and Dr. F. D. MURNAGHAN: Theory of functions of a real variable, three hours; Aerodynamics, three hours.—By Professor DANIELL: Statistics and probability, three hours.—By Professor EVANS: Differential and integral equations, three hours.—By Professor W. C. GRAUSTEIN: Projective geometry, three hours.

THE following is an additional list of university and college teachers of mathematics who have recently entered the national military service:

Major D. H. CRISSY, professor of mathematics at West Point, has been appointed commandant of the government school of aeronautics at Princeton. Professor C. W. COBB, of Amherst College, has been granted a year's leave of absence to enter the aviation work of the government. Professor W. J. BERRY, of the Brooklyn Polytechnic Institute, has been appointed second lieutenant 308th infantry, U. S. N. A. Professor E. I. SHEPARD, of Williams College, has been commissioned captain in the officers' reserve corps. Professor J. A. NYSWANDER, of the University of Nevada, Professor C. C. CAMP, of Ottawa University, Dr. L. M. KELLS, of the University of Illinois, and Dr. J. M. STETSON, of Adelbert College, have joined the national army. Dr. L. L. SILVERMAN, of Cornell University, has accepted a position with the Massachusetts board of public safety.

AT Purdue University associate professor WILLIAM MARSHALL has been promoted to a full professorship of mathematics, and Mr. GLENN JAMES has been promoted to an assistant professorship of mathematics. Professors G. ALFORD and J. WESTLUND have retired from active service.

AT the University of Michigan junior professor W. B. FORD has been promoted to a full professorship of mathematics, and Drs. L. A. HOPKINS and V. C. POOR have been promoted to assistant professorships of mathematics. Mr. G. H. CRESSE, Mr. G. D. JONES, and Dr. R. B. ROBBINS have been appointed instructors in mathematics. Dr. V. H. WELLS has been appointed professor of mathematics and statistics in the University of Pittsburgh, and Dr. A. L. MILLER has been appointed instructor in mathematics in Harvard University.

PROFESSOR C. N. HASKINS, of Dartmouth College, has been granted leave of absence for the present half-year.

AT the University of Pennsylvania Professor O. E. GLENN is absent on leave during the present half-year. Dr. J. R. KLINE has been appointed instructor in mathematics.

DR. HARRY BATEMAN has been appointed professor of aeronautical research and mathematical physics in Throop College.

AT the University of Texas adjunct professor J. W. CALHOUN has been promoted to an associate professorship of pure mathematics, and adjunct professor E. L. DODD has been promoted to an associate professorship of actuarial mathematics. Mr. H. J. ETTLINGER has been promoted to an adjunct professorship of applied mathematics. Dr. T. M. SIMPSON has been appointed to an adjunct professorship of mathematics.

MR. E. H. CLARKE, of Purdue University, has been appointed professor of mathematics in Hiram College.

DR. D. D. LEIB, of Connecticut College, has been promoted to an associate professorship of mathematics.

AT Iowa State College Dr. A. HELEN TAPPAN has been promoted to an assistant professorship of mathematics.

AT the Carnegie Institute of Technology at Pittsburgh Dr. J. C. MOREHEAD has been promoted to an assistant professorship of mathematics and descriptive geometry.

MR. I. L. MILLER, of Indiana University, has been appointed professor of mathematics in Carthage College.

DR. E. F. SIMONDS, of Columbia University, has been appointed instructor in mathematics in the University of Illinois.

DR. PAULINE SPERRY, of Smith College, has been appointed instructor in mathematics in the University of California.

MR. H. P. KEAN has been appointed instructor in mathematics in Milliken University.

DR. W. H. WILSON has been appointed instructor in mathematics in the Massachusetts Institute of Technology.

DR. G. W. SMITH has been appointed instructor in mathematics and acting head of the department of physics in Beloit College.

DR. C. C. BRAMBLE has been appointed instructor in mathematics in the U. S. Naval Academy.

DR. A. W. HOBBS has been appointed instructor in mathematics in the University of North Carolina.

DR. C. M. HEBBERT and DR. F. W. REED have been appointed instructors in mathematics in the aviation school of the University of Illinois.

DR. R. W. BURGESS, of Brown University, is now in the Bureau of Statistics at Washington.

DR. RUTH GENTRY died at Indianapolis, Ind., October 18, 1917. Miss Gentry had been a member of the American Mathematical Society since 1894.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOUTROUX (P.). See PASCAL (B.).

BRUNSCHVICG (L.). See PASCAL (B.).

BURKHARDT (H.). See ENCYCLOPÉDIE.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome II, volume 4, fascicule 2: Groupes de transformations continus; exposé, d'après l'article allemand de H. Burkhardt et L. Maurer, par E. Vessiot. Paris, Gauthier-Villars, 1916.

GAZIER (F.). See PASCAL (B.).

GEORGI (K.). Ueber die konforme Abbildung gewisser nichtsymmetrischer unendlichvielfach zusammenhängender Bereiche auf Kreisbereiche. (Diss.) Jena, 1915.

GMEINER (J. A.). See STOLZ (O.).

GROTENDORST (N. C.). Beginselen der differentiaal- en integraalrekening. Vierde verbeterde druk. Breda, De Koninklijke Militaire Academie, 1916. Gr. 8vo. 10 + 551 pp.

HARTMANN (F.). Zur Lösung des Fermatschen Problems. Leipzig, 1916. M. 1.00

KORTMULDER (R. J.). De logische grondslagen der wiskunde. (Diss.) Amsterdam, A. H. Kruyt, 1916.

MAURER (L.). See ENCYCLOPÉDIE.

MEROHICH (M.). Organistik der Geometrie. Grundzüge der geometrischen Prinzipienlehre, im Gegensatz zur euklidischen und nichteuklidischen Kasuistik. Moson, Ungarn, Selbstverlag, 1914. Gr. 8vo. 256 pp.

NABER (H. A.). Meetkunde en mystiek. Drie voordrachten. Amsterdam, N. V. Theosofische Uitgevers-Maatschappij, 1915. Gr. 4to. 99 pp.+7 plates. Fl. 4.50

PASCAL (B.). Oeuvres. Publiées suivant l'ordre chronologique avec documents complémentaires, introductions et notes par L. Brunschvicg, P. Boutroux, et F. Gazier. 14 tomes. Paris, Hachette, 1908–1914. 8vo. Chaque volume, Fr. 7.50

SCHUH (F.). Grepen uit de moderne meetkunde. (Noordhoff's Verzameling van Wiskundige Werken, Deel 2.) Eerste deel: Reciproke transformaties in het vlak en in de ruimte. Hyperboloïden en kegelsneden. Harmonische eigenschappen en cirkelbundels. Groningen, Noordhoff, 1916. Gr. 8vo. 24+590 pp.

STOLZ (O.) und GMEINER (J. A.). Theoretische Arithmetik. 2te Abteilung: Die Lehren von den reellen und von den komplexen Zahlen. 2te Auflage. Leipzig, Teubner, 1915. 8+369 pp. M. 12.00

VESSIOT (E.). See ENCYCLOPÉDIE.

II. ELEMENTARY MATHEMATICS.

ANDERSEN (L. P.), HØNHOLT (A.) og SCHMIDT (O.). Regnebog for Ungdomsskolen. København, Gyldensalske Boghandel, Nordisk Forlag, 1916. 24 pp.

BEHRENDSEN (O.) und GOTTING (E.). Lehrbuch der Mathematik nach modernen Grundsatzen. Ausgabe B fur Schulen mit realistischen Lehrplan. Oberstufe: Fur Oberrealschulen und Realgymnasien sowie für Studienanstalten realen Charakters. 2te Auflage. Leipzig, Teubner, 1916. 8vo. 7+404 pp.

COUNSEL for self-teachers. Treatise on methodology of mathematics pure and applied. (In Polish.) Vol. 1: Pure mathematics (20 monographs by S. Michalski, J. Lukasiewicz, Z. Janiszewski, S. Kwietniewski, W. Sierpinski, S. Zaremba, and S. Mazurkiewicz). Warsaw, A. Heflicha, 1915. 8vo. 39+618 pp.

CZERNY (A.). Die Erziehung zur Schule. (Schriften des Deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht. 2te Folge, Nr. 1.) Leipzig, Teubner, 1916. Gr. 8vo. 4+18 pp. M. 0.60

DEUTSCHER AUSSCHUSS. See CZERNY (A.), DIE TATIGKEIT, and TIMERDING (H. E.).

GÖTTING (E.). See BEHRENDSEN (O.).

HOČEVAR (F.). Lehr- und Uebungsbuch der Geometrie fur Gymnasien und Realgymnasien. 8te Auflage. Mittelstufe. Wien, Tempsky, 1916. K. 2.60

HØNHOLT (A.). See ANDERSEN (L. P.).

JANISZEWSKI (Z.). See COUNSEL.

KIRKESKOV (A. H. N.). Ungdomsskolens Regnebog 1–3. Regnetabeller og Regneregler samt en Bygningstegning. Løsninger til Fagopgaverne. København, Steen Hasselbalchs Forlag, 1916.

Klein (L.). Streifzüge in das Gebiet der Mathematik und Geometrie. Ein Hilfsbuch fur Lehrer und Schuler zur Belebung des Studiums und zur Benützung beim Selbstunterricht gesammelt. 3tes Heft: Ueber die Trisektion des Winkels. (Anhang über das sogenannte Delische Problem.) Korneuburg, Kuhkopf, 1915. K. 1.50

Kwietniewski (S.). See Counsel.

Lietzmann (W.). Geometrische Aufgabensammlung. Ausgabe A: Für Gymnasien. Leipzig, Teubner, 1916. 8vo. 8+174 pp. Geb. M. 2.00

Łukasiewicz (J.). See Counsel.

Lütken (J.). Regnebog for Seminarier. 1. Del. København, Lehmann og Stage, 1916. Kr. 2.00

Mazurkiewicz (S.). See Counsel.

Michalski (S.). See Counsel.

Schmidt (O.). See Andersen (L. P.).

Sierpinski (W.). See Counsel.

Die Tätigkeit des Deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht in den Jahren 1908–1913. Heft 1–18. Leipzig, Teubner, 1914. Gr. 8vo. 8+482 pp. Geh. M. 11.00; geb. M. 12.00

Timerding (H. E.). Die Aufgaben der Sexualpädagogik. (Schriften des Deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht. 2te Folge, Nr. 2.) Leipzig, Teubner, 1916. Gr. 8vo. 4+20 pp. M. 0.80

Zaremba (S.). See Counsel.

III. APPLIED MATHEMATICS.

Becker (H.). Geometrisches Zeichnen. Neubearbeitung von J. Vonderlinn. 3te Auflage. 3ter Neudruck. Berlin, 1915. M. 0.90

Beelenkamp (W. C.). See Grotendorst (N. C.).

Callendar (H. L.). See Moss (H.).

Doehlemann (K.). Grundzüge der Perspektive nebst Anwendungen. Leipzig, Teubner, 1916. M. 1.25

Dreyer (G.). Elemente der Graphostatik. Lehrbuch für technische Unterrichtsanstalten und zum Selbstunterricht, mit vielen Anwendungen auf den Maschinen- und Brückenbau. 4te Auflage. Ilmenau, 1916. M. 7.50

Dubash (P. S. G.). Continuity, or from electrons to infinity. Blackburn, G. Toulnim and Sons, 1917. 60 pp. 1s. 6d.

Dziobek (O.). Die Mechanik und ihre Anwendungen. Berlin, 1916. M. 3.00

Gans (R.). See Weber (R. H.).

Glazebrook (R. T.). Science and industry. The place of Cambridge in any scheme for their combination. The Rede Lecture, 1917. Cambridge, University Press, 1917. 51 pp. 1s. 6d.

Grotendorst (N. C.) and Beelenkamp (W. C.). Gronden der beschrijvende meetkunde. Eerste deel met aanhangsel. Tekst, Tiende druk.

Atlas, Tiende druk. Breda, De Koninklijke Militaire Academie, 1914. Gr. 8vo. 20+177 pp.+51 plates.

HERTZ (P.). See WEBER (R. H.).

ILLECK (J.). Richtiggestellte Theorie der Schwingungen gespannter Saiten nebst ihrer Anwendung auf zugehorige Probleme. Leipzig, 1916. M. 1.00

KAPTEYN (J. C.) and VAN UVEN (M. J.). Skew frequency curves in biology and statistics. Second paper. Groningen, Hoitsema, 1916. Gr. 8vo. 69 pp.

KRŽIWANEK (K.). Gravitation und Lichtgeschwindigkeit. Teschen, 1915. M. 4.00

MOSS (H.). Heat drop tables: absolute pressure. Calculated by H. Moss from the formulae and steam tables of H. L. Callendar. London, Arnold, 1917. 63 pp. 5s.

SCHWEYDAR (W.). Theorie der Deformation der Erde durch Flutkräfte. (Veroffentlichung des Kgl. Preussischen Geodätischen Instituts, Neue Folge, Nr. 66.) Leipzig, Teubner, 1916. 4to. 3+55 pp.

SEYFARTH (F.). Die Statik des Riemens. Ein Beitrag zur Mechanik linearer Kontinua. (Diss.) Jena, 1916.

SKUTSCH (R.). Zwei Vortrage über die Mechanik der Riementriebe, Dortmund, 1916. M. 1.50

SPRAGUE (E. H.). Elementary mathematics for engineers. 1917. For sale by Van Nostrand, New York. 236 pp. $1.50

TAYLOR (C. H.). Correction tables for thermodynamic efficiency. London, Arnold, 1917. 59 pp. 5s.

VAN UVEN (M. J.). See KAPTEYN (J. C.).

VONDERLINN (J.). See BECKER (H.).

WASHINGTON, Hydrographic office, U. S. Navy. Altitude, azimuth and line of position, comprising tables for working sight of heavenly body for line of position by the cosine-haversine formula, Marcq Saint Hilaire method and also Aquina's altitude and azimuth tables for line of position, Marcq Saint Hilaire method. (Hydrographic Office Publication, No. 200.) Washington, 1917. Sm. 4to. 319 pp.

WASSMUTH (A.). Grundlagen und Anwendungen der statistischen Mechanik. Braunschweig, 1915. M. 2.80

WEBER (R. H.) und GANS (R.). Repertorium der Physik. 1ter Band, 2ter Teil: Kapillaritat, Warme, Wärmeleitung, kinetische Gastheorie, statistische Mechanik (P. Hertz). Leipzig, Teubner, 1916. 8vo. 14+613 pp. Geb. M. 12.00

LIBRARY.

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and ninety-third regular meeting of the Society was held in New York City on Saturday, October 27, 1917. The attendance at the morning and afternoon sessions included the following thirty-five members:

Professor M. J. Babb, Dr. Emily Coddington, Professor F. N. Cole, Dr. W. L. Crum, Professor Louise D. Cummings, Professor L. P. Eisenhart, Professor P. F. Field, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. R. Hollcroft, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. R. Kline, Professor P. H. Linehan, Professor C. R. MacInnes, Professor H. B. Mitchell, Professor R. L. Moore, Dr. G. W. Mullins, Mr. L. S. Odell, Mr. George Paaswell, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Professor Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore.

Professor Veblen occupied the chair, being relieved by Professor Eisenhart. The Council announced the election of the following persons to membership in the Society: Dr. J. V. DePorte, State College, Albany, N. Y.; Mr. J. W. Lasley, Jr., University of North Carolina; Mr. Vicente Mills, Philippine Bureau of Lands; Professor B. M. Woods, University of California. Five applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. The Council reported a list of nominations for officers and other members of the Council to be printed on the official ballot for the annual election. The Secretary was directed to procure insurance to the extent of $10,000 on the library of the Society, now deposited in the Columbia University Library.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "Elementary inequalities for the roots of an algebraic equation."

(2) Professor LOUISE D. CUMMINGS: "The two-column indices for triad systems on fifteen elements."

(3) Dr. G. A. Pfeiffer: "On the continuous mapping of regions bounded by simple closed curves."

(4) Dr. J. F. Ritt: "On the differentiability of asymptotic series."

(5) Professor W. B. Fite: "Concerning the zeros of the solutions of certain linear differential equations."

(6) Professor J. E. Rowe: "Hexagons related to any plane cubic curve."

(7) Professor G. D. Birkhoff: "On a theorem concerning closed normalized orthogonal sets of functions with an application to Sturm-Liouville series."

(8) Professor Edward Kasner: "Systems of circles related to the theory of heat."

(9) Professor O. E. Glenn: "Systems of invariants and covariants of Einstein's transformations in the theory of relativity."

(10) Mr. J. K. Whittemore: "Theorems on ruled surfaces."

(11) Professor R. L. Moore: "On certain systems of equally continuous curves."

(12) Professor R. L. Moore: "Continua that have no continua of condensation."

(13) Dr. J. R. Kline: "Necessary and sufficient conditions, in terms of order, that it be possible to pass a simple continuous arc through a plane point set."

(14) Professor Oswald Veblen: "On the deformation of n-cells."

(15) Professor Oswald Veblen: "Deformations within an n-dimensional sphere."

Professor Birkhoff's paper was presented by Professor Dunham Jackson, and the papers of Professor Carmichael, Dr. Pfeiffer, Professor Rowe, and Professor Glenn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The principal object of Professor Carmichael's paper is to derive numerous elementary inequalities for the greatest absolute value X of a root of an algebraic equation. The methods employed are altogether elementary. The results generalize a number of known inequalities and contain several new ones of interest. In each case the inequality yields an upper bound to the value of X.

2. Professor Cummings' paper deals with the two-column indices as a method of comparison for triad systems. The two-column indices for the 80 systems on fifteen elements are obtained, and the supplemental theory necessary to make this method of comparison adequate is given. An important paper by Kirkman, which has apparently been overlooked by writers on triad systems up to the present time, is briefly discussed, and an inaccuracy in Netto's article on Tripelsysteme in the Encyklopädie der Mathematischen Wissenschaften is pointed out.

3. The theorem proved in Dr. Pfeiffer's paper is:

Given two simple plane curves J_1 and J_2 which are in $(1-1)$ continuous correspondence under a correspondence π. Let J_1 and J_2 be interior to the regions R_1 and R_2 respectively; then there exists a $(1-1)$ continuous correspondence between the points of R_1 and R_2 such that the points of J_1 correspond to points of J_2 and conversely as fixed by the correspondence π.

This theorem is essentially the theorem proved by Schoenflies in volume 62 (page 319) of the *Mathematische Annalen* to the effect that a $(1-1)$ continuous correspondence π between the interiors of two simple closed curves can be set up such that π is continuous with any given $(1-1)$ continuous correspondence between the boundaries (the given closed curves).

The important points of difference between the proof which Dr. Pfeiffer gives in this paper and that given by Schoenflies is first that the former is non-metric and second that no use is made therein of the Jordan theorem which states that a simple closed curve divides the plane into two regions. The Jordan theorem is thus an immediate corollary of the present proof when one of the curves J_1 is taken as a triangle and the corresponding R_1 as the whole plane.

4. Dr. Ritt shows that if $f(z)$ has, in a sector of the complex domain, the asymptotic development

$$f(z) \sim a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots,$$

then, in any sector interior to the given one,

$$f'(z) \sim a_1 + 2a_2z + \cdots + na_nz^{n-1} + \cdots.$$

This fundamental result was known and had been used by Professor Birkhoff, who failed to publish his proof, being under the impression that such a proof had already been given by Professor W. B. Ford.

5. Certain conditions under which the solutions of linear homogeneous differential equations of the second order vanish an infinite number of times are determined by Professor Fite in this paper. A theorem of comparison for the solutions of two such equations under circumstances different from those considered by Sturm results from one set of these conditions. Related questions in connection with certain binomial equations of the nth order are also considered.

6. In Professor Rowe's paper Theorems I and II of the second paper read by him before the Society at the Cleveland Meeting, September 4, 1917, which relate to hexagons inscribed in or circumscribed about the rational plane cubic curve, have been extended to apply to any plane cubic curve. The method used consists in expressing parametrically the coordinates of a point on the curve in terms of the elliptic functions $\wp(u)$ and $\wp'(u)$ (Durège-Maurer, Theorie der Elliptischen Functionen, pages 246–252), after which the proofs are algebraically as simple as in the case of the rational plane cubic curve.

7. The theorem of Professor Birkhoff's note states that if a set of normalized orthogonal functions $u_1(x)$, $u_2(x)$, $\cdots$ $(0 \leqq x \leqq 1)$ is closed, and if $\bar{u}_1(x)$, $\bar{u}_2(x)$, $\cdots$ is a second set of normalized orthogonal functions such that the infinite series $\sum_{n=1}^{\infty} (u_n(x) - \bar{u}_n(x))u_n(y)$ converges to a function $H(x, y)$ less than 1 in absolute value, then, under a certain simple further condition, the second set $\bar{u}_1(x)$, $\bar{u}_2(x)$, $\cdots$ is also closed.

By means of this theorem it is readily proved that any set of characteristic functions of Sturm-Liouville type is closed. An elementary and simple proof of this important fact has hitherto been lacking.

The theorem is really one in the field of general analysis as defined by E. H. Moore and can be extended to biorthogonal sets of functions.

The note will appear in the *Proceedings of the National Academy of Sciences.*

8. In the propagation of heat in a plane there are in general ∞^2 isothermal curves, ∞^1 for each value of the time. Professor Kasner shows that these ∞^2 curves can never be circles. If, however, the system degenerates into ∞^1 curves (that is, if the curves are the same at all times), as discussed in an earlier paper, systems of circles are possible, and all such cases are determined.

9. Professor Glenn's paper deals with complete invariant systems under the Einstein formulas S for the transformation of space and time coordinates. Systems are derived for binary forms in t and x (t being the time), with coefficients constant or arbitrary functions of the quantities left fixed by S. Their general theory is that of concomitants which are functions of the coefficients of the transformation, but they are, in fact, free from the relative velocity of the moving systems of reference, while involving the constant c, the velocity of light. Included in the paper is the explicit system of relative concomitants for the form of general order m, and absolute systems for the orders to the fourth inclusive.

10. Mr. Whittemore's paper is a continuation of that presented to the Society at the April meeting. It is concerned chiefly with the determination of the "flat points" of any ruled surface, and with the discussion of the properties of orthogonal trajectories of the rulings at points where the total curvature of the surface vanishes.

11. Suppose that G is a set of simple continuous arcs each of which has one end point on the side AD and the other end point on the side BC of the square $ABCD$. Suppose that each arc of the set G lies, except for its end points, entirely within $ABCD$, that no two arcs of G have any point in common and that every point within or on $ABCD$ belongs to some arc of the set G. In this paper Professor Moore shows that a necessary and sufficient condition that such a set of arcs G may be mapped continuously on a set of straight segments possessing these same properties is that the arcs of the set G should be equally continuous.

12. In this paper Professor Moore shows that in two-dimensional space every continuum that has no continuum

of condensation is a continuous curve. The theorem that every such continuum which is irreducible between two points is a simple continuous arc is due to S. Janiszewski. Cf. his paper "Sur les continus irréductibles entre deux points," *Journal de l'Ecole Polytechnique*, 2e série, volume 16 (1911–12), pages 79–170.

13. Dr. Kline proves that, in order that it may be possible to pass at least one simple continuous arc through a given bounded closed point set K, it is both necessary and sufficient that the points of K may be ordered subject to the following conditions:

(1) If $A \neq B$, then either A precedes B or B precedes A.

(2) If A precedes B, then B does not precede A.

(3) If A precedes B and B precedes C, then A precedes C.

(4) If the point P of K is a geometrical limit point of some subset M of K, then every segment of K containing P (or every segment one of whose end points is P, if P is either the first or last point of K) contains at least one point of M, different from P.

If K is not closed, these conditions are not sufficient. In this case, he proves that a necessary and sufficient set of conditions is obtained if the points of the open set K can be ordered subject to conditions (1) to (4) and the following two additional conditions:

(5) If $A_1, A_2, A_3, \cdots$ is a countable infinity of distinct points of K such that either A_n precedes A_{n+1} for all values of n or A_{n+1} precedes A_n for all values of n, then the point set $A_1, A_2, A_3, \cdots$ has a single limit point A.

(6) If $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ are ordered sequences, approaching A and B, respectively, such that, for all values of n and k, A_n precedes B_k, then, if $A = B$, A_n precedes A_{n+1} $(n = 1, 2, 3, \cdots)$, B_{k+1} precedes B_k $(k = 1, 2, \cdots)$ and there is no point D $(D \neq A)$ such that D follows every A and precedes every B.

14. Professor Veblen's first paper contains a proof of the theorem that any $(1-1)$ continuous transformation of an n-cell and its boundary into themselves, which leaves all points of the boundary invariant, is a deformation within an n-dimensional euclidean space containing the n-cell.

15. In Professor Veblen's second paper it is proved that there does not exist a deformation on an n-sphere which carries each point of an $(n - 1)$-sphere into itself and one of the n-cells bounded by the $(n - 1)$-sphere into the other.

F. N. Cole,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

The thirtieth regular meeting of the San Francisco Section was held at the University of California, on Saturday, October 27. The attendance included the following seventeen members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor L. E. Dickson, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam, Dr. Pauline Sperry, Mr. J. S. Taylor.

Professor Manning and Dr. Bernstein were elected chairman and secretary, respectively, for the ensuing year. Professors Lehmer and Blichfeldt and Dr. Bernstein were elected members of the programme committee.

The next meeting of the Section will be held at Stanford University, April 6, 1918. The succeeding fall meeting will be held at the University of California on October 26.

The following papers were presented at this meeting:

(1) Dr. B. A. Bernstein: "On a numero-logical foundation for the theory of probability."

(2) Professor L. E. Dickson: "Some unsolved problems in the theory of numbers."

(3) Professor L. M. Hoskins: "The strain of a gravitating, compressible sphere of variable density."

(4) Professor W. A. Manning: "On the order of primitive groups, IV."

(5) Professor R. M. Winger: "The rational plane cubic as an application of the theory of involution."

Professor Winger's paper was read by title. Abstracts of the papers follow in the order given above.

1. Dr. Bernstein proposes a set of postulates for the theory of probability based on Boolean logic.

2. With due attention to historical perspective, Professor Dickson discussed various unsolved problems in the theory of numbers, including questions on perfect numbers, amicable numbers of higher order, symmetric functions of the totitives of a composite modulus, Fermat's numbers connected with inscriptible regular polygons, Waring's problem, and the distribution of primes. In particular, the opinion was expressed that various topics in the theory of numbers presented an excellent opening for investigation both by amateurs and professional mathematicians who are not actively engaged upon mathematical investigations. No separate publication of this paper will be made, as the material presented is to be found in the writer's History of the Theory of Numbers now in press by the Carnegie Institution.

3. The problem of the strain of a gravitating sphere by small disturbing forces having a spherical harmonic potential has been completely solved for the case of uniform density, and certain cases of variable density have been solved on the assumption of incompressibility. Professor Hoskins's paper contains the solution for an important class of cases of variable density without restriction of the value of the modulus of compression. The cases covered are those in which the density is a rational integral function of the distance from the center. Reasonable assumptions regarding the density of the earth may be represented to a close approximation by relatively simple functions of this kind. A series of numerical results has been obtained for a case in which the density function is a binomial.

4. In this sequel to former studies in primitive groups it is shown that if a primitive group, not alternating or symmetric, contains a substitution of prime order p and of degree $6p$, $p > 6$, its degree in no case exceeds $6p + 10$. In fact the degree cannot be $6p + 7$ or $6p + 8$ if $p > 7$. If $p = 7$, the limit of the degree is 49. Professor Manning found his task considerably lightened by the following theorem: A simply transitive primitive group in whose subgroup that leaves one letter fixed there is one and only one doubly transitive con-

stituent of degree m must have in this maximal subgroup at least one transitive constituent the degree of which is a divisor ($> m$) of $m(m - 1)$. This paper has been offered to the *Transactions* for publication.

5. In this paper Professor Winger shows how the classical properties of the rational cubic, R^3, can be derived quite simply from the theory of involution. The method is then employed in the discovery of new theorems. In particular the contact conics, including the perspective conics, are discussed. The paper closes with some theorems on the hyperosculating curves, i. e., curves whose complete intersections with R^3 fall at a point.

W. A. MANNING,
Secretary of the Section.

ON INTEGRALS RELATED TO AND EXTENSIONS OF THE LEBESGUE INTEGRALS.

BY PROFESSOR T. H. HILDEBRANDT.

(*Continued from page 144*)

III. STIELTJES INTEGRALS AND THEIR GENERALIZATIONS.

While the Lebesgue integral received almost immediate attention and recognition and found its way rapidly into mathematical literature and thought, it is only recently that the definition of Stieltjes seems to have received the consideration to which it is entitled by virtue of its range of applicability and usefulness. As a matter of fact, in the opinion of the writer, it seems to be destined to play the central rôle in integrational and summational processes in the future.

1. *Definition of the Stieltjes Integral.*—(Cf. Stieltjes (23), pages 71 ff.; Perron (17), page 362; Fréchet (5), pages 45–54; Young (29), pages 131, 137.) A definition for this integral was given first by Stieltjes in his memoir on continued fractions. The integral depends for its value upon two functions $f(x)$ and $v(x)$ defined on an interval (a, b). We suppose that they are both bounded. Then the definition is as follows:

DEFINITION. *Divide* (a, b) *into a finite number of intervals by the points* $a = x_0, x_1, \cdots, x_n = b$. *Let* ξ_i *be any point interior to* (x_{i-1}, x_i) *and form the sum*

$$s = \sum_{i=1}^{n} f(\xi_i)(v(x_i) - v(x_{i-1})).$$

If this sum has a limit as the number of divisions is increased and their maximum length diminished, this limit is the Stieltjes integral

$$\int_a^b f(x)dv(x).$$

Stieltjes stated his definition for $f(x)$ any continuous function and $v(x)$ a monotonic non-decreasing function and showed that in this case the integral exists. If we desire this integral to exist for every function continuous on (a, b) it is necessary and sufficient that $v(x)$ be of bounded variation. However for the existence of an $\int fdv$ it is not necessary either that $f(x)$ be continuous nor that $v(x)$ be of bounded variation. For instance, we have the proposition

(1) If $f(x)$ is continuous and $v(x)$ of bounded variation then the Stieltjes integral

$$\int_a^b vdf$$

exists also and we have

$$\int_a^b vdf = \int_a^b fdv + f(b)v(b) - f(a)v(a).$$

The proof depends upon the rearrangement of the terms in the sum s.*

In the sequel we shall confine ourselves mainly to the case in which v is a function of bounded variation, or more particularly monotonic non-decreasing, from which the former case can be deduced on account of the fact that every function of bounded variation can be expressed as the difference of two monotonic non-decreasing functions.

As in the case of a Riemann integral we have the following theorem relative to the existence of a Stieltjes integral, v being a function of bounded variation:

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* Cf., for instance, Bliss (1), p. 29.

(2) *A necessary and sufficient condition for the existence of the Stieltjes integral* $\int f dv$ *is that the total variation* of the function* $v(x)$ *over the points at which* $f(x)$ *is discontinuous shall be zero.*

Among other things, this theorem would require that the function v be continuous at points of discontinuity of f.

The following simple instances of Stieltjes integrals may be of interest:

(*a*) If there exists a function $w(x)$ such that

$$v(x) = \int_a^x w(x)dx,$$

then

$$\int_a^b f(x)dv(x) = \int_a^b f(x)w(x)dx,$$

the integral on the right-hand side being taken in the same sense as that of $w(x)$.

(*b*) Suppose $v(x)$ monotonic non-decreasing, and discontinuous at the points $x_1 \leqq x_2 \leqq x_3 \cdots$, which approach b as a limit; let the measure of the discontinuities of $v(x)$ at these points be the positive numbers $v_1, v_2, \cdots$, respectively. Of necessity the series $\Sigma_n v_n$ is convergent. In the interior of the interval $(x_{n-1} \leqq x < x_n)$ let $v(x)$ be constant and equal to $\sum_{i=1}^{n-1} v_i$, and $v(b) = \Sigma_n v_n$. Then evidently, if $f(x)$ is continuous in (a, b),

$$\int_a^b f(x)dv(x) = \Sigma_n v_n f(x_n).$$

(*c*) If in example (*b*) we assume that in the interval $(x_{n-1} \leqq x < x_n)$, we have

$$v(x) = x + \sum_{i=1}^{n-1} v_i,$$

with $v(b) = b + \Sigma_n v_n$, then

$$\int_a^b f(x)dv(x) = \int_a^b f(x)dx + \Sigma_n v_n f(x_n).$$

These examples illustrate the fact that the Stieltjes integral in addition to giving an ordinary definite integral introduces also the values of the function integrated at special points. In this direction Fréchet has shown

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* For the definition of the total variation over a set of points cf. below, § 5. This theorem has been communicated to me by Bliss. Cf. also Young (29), pp. 132, 133, for the case where $v(x)$ is monotonic.

(3) The Stieltjes integral $\int_a^b f(x)dv(x)$ can be broken up into the sum of three terms

$$\int_a^b f(x)dv(x) = \int_a^b f(x)a(x)dx + \Sigma_n v_n f(x_n) + \int_a^b f(x)du(x),$$

in which $a(x)$ is a summable function and hence the first term a Lebesgue integral; the x_n are the points of discontinuity of $v(x)$ and $v_n = v(x_n + 0) - v(x_n - 0)$; and u is a continuous function of bounded variation which has a derivative zero excepting at a set of measure zero.

For if we set

$$\varphi(x) = \sum_{a<x_n\leqq x} (v(x_n) - v(x_n - 0)) + \sum_{a\leqq x_n<x} (v(x_n + 0) - v(x_n)),$$

then the function $v(x) - \varphi(x)$ is continuous and of bounded variation. Hence, if $a(x)$ is equal to one of its derived functions wherever this is finite, and zero everywhere else, then

$$v(x) - \varphi(x) = \int_a^x a(x)dx + u(x),$$

where $u(x)$ is the variation of $v(x) - \varphi(x)$ over the set of points of measure zero at which the derivative $a(x)$ would be infinite.* The form of the theorem is then an immediate consequence of this decomposition of the function $v(x)$.

2. *Some Properties of the Stieltjes Integral.*—(Cf. Perron (17), pages 366 ff.; Riesz (21), pages 38, 39.) We note the following properties of the Stieltjes integral, v being a function of bounded variation and f being supposed to be integrable with respect to v in the interval (a, b):

(1) $\int_a^c fdv + \int_c^b fdv = \int_a^b fdv.$

(2) $\int_a^b (f + g)dv = \int_a^b fdv + \int_a^b gdv.$

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* Cf. Bliss (1), p. 40; de la Vallée Poussin (24), p. 93.

(3) $\int_a^b cfdv = c\int_a^b fdv.$

(4) If $f \geqq 0$ and $v(x)$ is monotonic non-decreasing then

$$\int_a^b fdv \geqq 0.$$

(5) $\left|\int_a^b fdv\right| \leqq \int_a^b |f|\,|dv| = \int_a^b |f|du,$ where u is the total variation of $v(x)$ from a to x.

(6) $\int_a^b dv = v(b) - v(a).$

(7) If $\lim_n f_n(x) = f(x)$ uniformly on (a, b), then

$$\lim_n \int_a^b f_n(x)dv = \int_a^b f(x)dv.$$

(8) *If* $\int_a^b fdv = 0$ *for all continuous functions and* v *is a function of bounded variation, then* $v(x)$ *is constant except at a denumerable set of points between* a *and* b, *and conversely.*

3. *Comparison of the Stieltjes Integral and the Lebesgue Integral.*—(Cf. Lebesgue (14); Van Vleck (25).) Lebesgue (14) has shown that every Stieltjes integral is expressible as a Lebesgue integral. He gives two modes of procedure. In the first let $w(x)$ be the total variation of $v(x)$ in the interval (a, x). Let $x(w)$ be the inverse function of w, with the convention that in case $w(x)$ is constant in the interval (l, m), we take for $x(w)$ any of the values in this interval, for instance l. Substitute this function in the function $v(x)$, and assume that if x_0 is a point of discontinuity of $v(x)$ then we make $v(x(w))$ linear in the intervals between $w(x_0 - 0)$ and $w(x_0)$, and $w(x_0)$ and $w(x_0 + 0)$. Then $v(x(w)) = u(w)$ will be a function of bounded variation having total variation equal to w at any point, and so, if u' is any of the derivatives of u with respect to w, it will take only the values $+ 1$ and $- 1$, and we shall have

$$u(w) = \int_0^w u'(w)dw.$$

Hence

$$\int_a^b f(x)dv(x) = \int_0^{w(b)} f(w)u'(w)dw,$$

where now $f(w)$ is no longer continuous but has a denumerable set of discontinuities of the first kind, i. e., $f(w_0 - 0)$ and $f(w_0 + 0)$ exist at every point; and the integral on the right is a Lebesgue integral.

The other transformation which Lebesgue suggests rests upon the fact that a function of bounded variation can be expressed as the difference of two monotonic non-decreasing functions. If $v(x) = g(x) - h(x)$ and we define $x(g)$ and $x(h)$ as above, and then replace the variables g and h by

$$g - g(a) = (g(b) - g(a))t = K_1 t,$$
$$h - h(a) = (h(b) - h(a))t = K_2 t$$

and set $x(g) = x_1(t)$ and $x(h) = x_2(t)$, then we can express

$$\int_a^b f(x)dv(x) = \int_0^1 [K_1 f(x_1(t)) - K_2 f(x_2(t))]dt.$$

We note that in the Fréchet expression of § 1, the interval of variation of f is still (a, b); in both of these last expressions we have a new variable of integration and a new interval of integration.

On the other hand Van Vleck (25) has given a very simple transformation of a Lebesgue integral into a Stieltjes integral. As a matter of fact we might point out that the *Lebesgue integral is by its very manner of definition a Stieltjes integral.*

He shows that if $l < f(x) < L$ and $\mu(y)$ is defined to be the measure of the set E for which $l \leqq f < y$, then

$$(L)\int_a^b f(x)dx = (S)\int_l^L y d\mu(y).$$

By using the integration by parts formula (1), Bliss ((1), page 28) shows that a Lebesgue integral is reducible to a Riemann integral,* viz.,

$$(L)\int_a^b f(x)dx = (S)\int_l^L y d\mu(y) = L(b - a) - (R)\int_l^L \mu(y)dy.$$

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* Cf. also Young (26), pp. 245ff.; Denjoy (4(b)), pp. 191–203; Lamond, *American Journal of Mathematics*, vol. 36 (1914). pp. 387, 388, in which the final formulas are incorrect.

If $f(x)$ is not bounded on (a, b), we have

$$(L) \int_a^b f(x)dx = \int_{-\infty}^{\infty} y d\mu(y),$$

where

$$\int_{-\infty}^{\infty} y d\mu(y) = \lim_{\alpha,\, \beta \doteq \infty} \int_{-\alpha}^{\beta} y d\mu(y).$$

A similar reduction of the Pierpont integral to the Stieltjes, and thence to a Riemann integral can be made, if we express the measure function $\mu(y)$ in terms of upper measure instead of measure.

4. *Applications.*—(Cf. Riesz (21), pages 40–43, 51–54; (22).) We would seem to have shown in the last section that the Stieltjes and Lebesgue integrals are equivalent. This is true in the sense that each of them can by a transformation be evaluated in terms of the other, but in the transformation we change both the function and the interval of integration. It does not however follow from this that wherever we use a Stieltjes integral a Lebesgue integral will serve just as well. Perhaps the best illustration of this fact are the two applications of Stieltjes integrals which we shall briefly consider, the first of which has probably contributed more than any other to direct attention to the Stieltjes integral.

The first application is in the theory of linear functional operations. Let $\mathfrak{F}$ be the class of all continuous functions f on an interval (a, b). Then U is said to be a functional operation on $\mathfrak{F}$, if for every function f of $\mathfrak{F}$ there is a corresponding real number $U(f)$.

A functional operation which is *linear* is generally defined to have the following two properties:

(*a*) *distributivity*: $U(f_1 + f_2) = U(f_1) + U(f_2)$;

(*b*) *continuity*: if $\lim_n f_n = f$ uniformly on (a, b), then $\lim_n U(f_n) = U(f)$.

It can then be shown that every linear functional operation U possesses also the properties

(*c*) $U(cf) = cU(f)$, c being any constant;

(*d*) there exists a quantity M such that, for every function f of the class $\mathfrak{F}$, $|U(f)| \leqq M \times$ maximum of $|f|$.

If the functional operation U has the property (*a*), then the property (*d*) is equivalent to the continuity property (*b*).

As illustrations of linear functional operations we cite

$$U(f) = \int_a^b f(x)u(x)dx,$$

$$U(f) = \Sigma_n u_n f(x_n),$$

where $u(x)$ is any summable function on (a, b), and where the u_n constitute an absolutely convergent series and the x_n are any points of the interval (a, b).

Hadamard* has shown that for every linear functional operation $U(f)$ on the class of all continuous functions there exists a sequence of integrable functions $u_n(x)$ such that

$$U(f) = \lim_n \int_a^b f(x)u_n(x)dx,$$

but the functions $u_n(x)$ do not necessarily have a summable function as a limit. On the other hand, Helly† has shown that there exist a set of constants $u_m{}^{(n)}$, a set of points $x_m{}^{(n)}$ on the interval (a, b), and a set of integers r_m, such that

$$U(f) = \lim_{m;\, r_m = \infty} \sum_{n=1}^{r_m} u_m{}^{(n)} f(x_m{}^{(n)}).$$

Each of these expressions for a linear $U(f)$ is rather complicated. Perhaps the most elegant expression for a linear operation has been given by Riesz, who shows that there exists a function of bounded variation $u(x)$ such that

$$U(f) = \int_a^b f(x)du(x).$$

Riesz's second proof of this fact (cf. (22)) is really quite simple and elegant. It is made to depend upon the extension of the application of $U(f)$ to functions which are constant on intervals, and this leads directly to a definition of the function $u(x)$ and to the expression of $U(f)$ for any continuous function $f(x)$ as a Stieltjes integral.

A second application is allied to the Weierstrass theorem that every continuous function is expressible as the limit of a uniformly convergent sequence of polynomials, i. e., linear combinations of the functions $1, x, x^2, \cdots, x^n, \cdots$. This theorem has suggested the question: under what conditions, necessary and sufficient, can we say of a given set of functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$, continuous on (a, b), that any continuous function can be expressed as the limit of a uniformly convergent sequence of functions which are linear combinations of the $\varphi_n(x)$.

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* Cf. Leçons sur le Calcul des Variations, I, pp. 289–302.

† Cf. *Wiener Berichte*, vol. 121_{2a} (1912), pp. 277–8.

Various necessary or sufficient conditions in terms of Riemann integrability and Lebesgue summability were given, involving the derivatives of the functions φ_n, but it remained for the Stieltjes integral to provide a condition both necessary and sufficient, which is as follows (cf. Riesz (21), pages 51–54):

A necessary and sufficient condition that every continuous function may be uniformly approximated by linear combinations of a set of functions $[\varphi_1(x), \cdots, \varphi_n(x), \cdots]$ is that the only solution of the equations

$$\int_a^b \varphi_n(x)du(x) = 0 \qquad (n = 1, 2, \cdots),$$

for a $u(x)$ which for every x_0 satisfies the condition

$$u(x_0) = \tfrac{1}{2}(u(x_0 + 0) + u(x_0 - 0)),$$

is $u(x) =$ constant.

5. *Extension of the Stieltjes Integral.*—In constructing a descriptive definition of integration Lebesgue* sets down the following six properties sufficient to characterize his definition of integration:

$L(1)$ $\displaystyle\int_a^b f(x)dx = \int_{a+h}^{b+h} f(x-h)dx.$

$L(2)$ $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^a f(x)dx = 0.$

$L(3)$ $\displaystyle\int_a^b f(x)dx + \int_a^b \varphi(x)dx = \int_a^b (f(x) + \varphi(x))dx.$

$L(4)$ If $f \geqq 0$ and $b > a$, then $\displaystyle\int_a^b f(x)dx \geqq 0.$

$L(5)$ $\displaystyle\int_0^1 1 \times dx = 1$ or $\displaystyle\int_a^b 1 \times dx = b - a.$

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* Cf. (13), pp. 98, 99. Cf. also *Annales de l'Ecole Normale Supérieure*, ser. 3, vol. 27 (1910), pp. 368, 369 and 374 ff.

$L(6)$ If $f_n(x)$ is a monotonic non-decreasing sequence of functions having $f(x)$ as a limit, then

$$\lim_n \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

If we turn back to § 2, we observe that there are properties of the Stieltjes integral which are very similar to these; in particular we find a marked similarity between (1), (2), (4) and (6) of § 2 and $L(2)$, $L(3)$, $L(4)$, and $L(5)$ above. We observe that the main dissimilarity between the properties are in the absence of an analogue of $L(1)$ and the fact that in the analogue (7) of $L(6)$ the convergence to the limiting function $f(x)$ is uniform instead of monotonic. In discussing this last difference first, we observe that it is really *the property* $L(6)$ *which characterizes the Lebesgue integral as distinct from the Riemann,* i. e., in effect that the continuity of the Riemann integral operation is one based on uniform convergence of the functions, and that of the Lebesgue one is based on monotonic sequences. It is this distinction which we find emphasized in the later treatments of Young (cf. (27) and (28)). It seems that an obvious conclusion is that, if we start from the following properties modelled after those of Lebesgue, we might expect to arrive at a Lebesgue generalization of the Stieltjes integral:

$S(1)$ $v(x)$ is a monotonic non-decreasing function of x.

$S(2)$ $\int_a^b f(x)dv(x) + \int_b^c f(x)dv(x) + \int_c^a f(x)dv(x) = 0.$

$S(3)$ $\int_a^b (f(x) + \varphi(x))dv(x) = \int_a^b f(x)dv(x) + \int_a^b \varphi(x)dv(x).$

$S(4)$ If $f \geqq 0$ and $b > a$, then $\int_a^b f(x)dv(x) \geqq 0.$

$S(5)$ $\int_a^b dv(x) = v(b) - v(a).$

$S(6)$ If f_n is a monotonic non-decreasing sequence of functions such that $\lim_n f_n = f$, then

$$\lim_n \int_a^b f_n(x)dv(x) = \int_a^b f(x)dv(x).$$

In the above properties $S(1)$ is not an analogue of $L(1)$, which would have the form

$$\int_a^b f(x)dv(x) = \int_{a+h}^{b+h} f(x-h)dv(x).$$

If we use this property in connection with $S(5)$ we get

$$v(b') - v(a') = v(b'') - v(a'')$$

for every pair of subintervals of (a, b) for which

$$b' - a' = b'' - a''.$$

In as much as v is a continuous function except at a denumerable infinity of points, this would require that the function $v(x)$ be continuous throughout the interval, and from this in turn we would conclude that $v(x)$ is linear and of the form $cx + d$. If we impose on this the requirement $L(5)$ instead of $S(5)$ we get $v(x) = x + c$. In other words, it seems that the property $L(1)$ plays a principal rôle in characterizing the Lebesgue and Riemann integrals as distinct from the Stieltjes.

By following through a method of reasoning similar to that of Lebesgue ((13), pages 98–102), but on the basis of the properties $S(1)$–$S(6)$, we arrive at a result similar to the one which he obtains, viz.,

In order to find $\int_a^b fdv$, it is sufficient to know how to find $\int_a^b \psi dv$, where ψ is a function which takes only the values zero and unity.

Lebesgue determines his function ψ to be the measure of the set of points E for which $\psi = 1$. Radon ((19), pages 1305 ff.) has suggested a method of procedure to be applied to any monotonic non-decreasing function which has the continuity property $v(x-0) = v(x)$ at every point of the interval. This requirement is a result of the fact that he considers as the basis of his operations half open intervals $(a \leqq x < b)$. We modify this method slightly, using open intervals instead of half open ones.

We proceed to define a function $v(E)$ on a set of points E corresponding to the measure function, as follows: If $E = (a' < x < b')$ then $v(E) = v(b' - 0) - v(a' + 0)$. If a and b are the extremities of the fundamental interval, then we take for $E = (a \leqq x \leqq b)$, $v(E) = v(b) - v(a)$. In order to avoid circumlocution, we might think of the interval (a, b) as extended by ϵ on each end, and have $v(a - \epsilon) = v(a)$ and $v(b + \epsilon) = v(b)$, and think of the interval (a, b) as enclosed in an open interval extending beyond a and b. Complementary sets will be taken with respect to the *closed* interval (a, b).

For any set E, we define then a quantity $v(E)$ as follows: Enclose the points of E in a finite or denumerable infinitude of open intervals α_n; then

$$\bar{v}(E) = \bar{B}(\Sigma_n v(\alpha_n)),$$

i. e., $\bar{v}(E)$ is the least upper bound of the sums $\Sigma_n v(\alpha_n)$ for all possible enclosures of the set E in open intervals. Let CE be the complementary set to E relative to the closed interval (a, b). Then we define

$$\underline{v}(E) = v(a, b) - \bar{v}(CE).*$$

When $\bar{v}(E) = \underline{v}(E)$, we say that E is *measurable relative to* $v(x)$, and define $v(E)$ to be the common value. The totality $\mathfrak{E}$ of sets measurable relative to $v(x)$ form a class of sets which have the following properties:†

(1) If E_1 and E_2 belong to the class $\mathfrak{E}$, then $E_1 + E_2$ and E_1E_2 also belong to $\mathfrak{E}$.

(2) If $E_1, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$ and are mutually distinct, then $\Sigma_n E_n$ also belongs to $\mathfrak{E}$.

(3) The Borel measurable sets belong to $\mathfrak{E}$.

Further $v(E)$ has the properties

(1) If E_1 and E_2 belong to $\mathfrak{E}$, then

$$v(E_1 + E_2) + v(E_1E_2) = v(E_1) + v(E_2).$$

(2) If $E_1, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$ and are mutually distinct, then

$$v(\Sigma_n E_n) = \Sigma_n v(E_n),$$

i. e., v is what Radon ((19), page 1299) calls an *absolutely additive function.*

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* Note that the values of $\bar{v}(E)$ and $\underline{v}(E)$ are independent of the values of $v(x)$ at its points of discontinuity. We might then have assumed with Radon $v(x - 0) = v(x)$ at every point. In particular we have

$$v(E \equiv a' \leqq x \leqq b') = v(b' + 0) - v(a' - 0), \text{ not } v(b') - v(a').$$

† For the proof cf. Radon (19), pp. 1305 ff. Bliss (1), pp. 12–17, has given a careful analysis of the case in which $v(x)$ is a continuous monotonic non-decreasing function.

(3) For every E of $\mathfrak{E}$ and for every $\epsilon > 0$, there exists a closed set E', contained in E and $\mathfrak{E}$, such that $v(E) - v(E') < \epsilon$.

In case $v(x)$ is of bounded variation instead of monotonic non-decreasing, it can be expressed as the difference of two monotonic non-decreasing functions $p(x)$ and $n(x)$. We can then find the class $\mathfrak{E}_1$ of all sets measurable relative to $p(x)$ and $\mathfrak{E}_2$ of all sets measurable relative to $n(x)$. We say that the class $\mathfrak{E}$ of all sets measurable relative to v is the greatest common subclass of the classes $\mathfrak{E}_1$ and $\mathfrak{E}_2$, and set

$$v(E) = p(E) - n(E).$$

The *total variation of v over the set E* is defined to be

$$t(E) = p(E) + n(E).$$

6. *The Fréchet Generalized Integral.*—(Cf. Fréchet (6).) Instead of proceeding to the definitions of integration of the function f with respect to the $v(E)$ defined in the preceding section, on an interval (a, b) or a set E, we prefer to discuss the definition of integration suggested by Fréchet, which includes the Lebesgue, Young, Pierpont, and Stieltjes integrals as special cases by properly assigning the function v.

Suppose a set $\mathfrak{P}$ of general elements p. Suppose further a class $\mathfrak{E}$ of subsets E of elements of p, which has the following properties:

(1) If E_1 and E_2 belong to $\mathfrak{E}$, then $E_1 + E_2$ and E_1E_2 also belong to $\mathfrak{E}$.

(2) If $E_1, \cdots, E_n, \cdots$ are sets belonging to $\mathfrak{E}$, which are without common elements, then $\Sigma_n E_n$ also belongs to $\mathfrak{E}$.

On the class $\mathfrak{E}$, we suppose that there is defined an *absolutely additive function* $v(E)$ which has the following property: If $E_1, \cdots, E_n, \cdots$ are sets belonging to $\mathfrak{E}$ without common elements, then

$$v(\Sigma_n E_n) = \Sigma_n v(E_n).$$

Then it can be shown that there exist two functions $v_1(E)$ and $v_2(E)$, satisfying the same condition as v and in addition the condition $v_1(E) \geqq 0$ and $v_2(E) \geqq 0$ for every E, and such that

$$v(E) = v_1(E) - v_2(E).$$

We shall assume in the sequel that in addition to being absolutely additive, the function $v(E)$ is also *monotonic*, i. e.,

$$v(E) \geqq 0 \text{ for every } E.$$

The changes in the properties and definitions given below are obvious, when this last requirement is omitted.

Suppose there is defined on $\mathfrak{P}$ a function $f(p)$. In defining a value for an integral $\int_E f dv$, we can follow the method of Young or that of Lebesgue.

For an analogue of the Young integration, we divide E into a finite or denumerable set of subsets E_n of $\mathfrak{E}$, and let m_n and M_n be the upper and lower bounds, respectively, of f on E_n. Let

$$S = \Sigma_n M_n v(E_n) \quad \text{and} \quad s = \Sigma_n m_n v(E_n).$$

Then the upper integral $\overline{\int}_E f dv$ is the greatest lower bound of the values of S and $\underline{\int}_E f dv$ the least upper bound of the values of s for all possible divisions of E into a finite or denumerably infinite number of sets belonging to $\mathfrak{E}$. We say that $\int_E f dv$ exists, when the upper and lower integrals are equal.

If f is not bounded on E, then the divisions of E must be restricted to be such that M_n and m_n are finite. In particular a necessary and sufficient condition for the existence of finite values for $\overline{\int}$ and $\underline{\int}$, the upper and lower integrals, is that there shall be at least one division of E into a denumerable set of distinct sets E for which the value of S formed for $|f|$, $\Sigma_n M_n v(E_n)$, is finite.

On the other hand we are able to give a definition of integration which is analogous to that of Lebesgue. We say that f is *measurable relative to* $\mathfrak{E}$, if the set E for which $f > l$ belongs to $\mathfrak{E}$ for every value of l. It follows from this that the sets E for which $l_1 \leqq f < l_2$ belong to $\mathfrak{E}$ for every l_1 and l_2.

Let us divide the interval $-\infty$ to $+\infty$ by means of the points $\cdots, l_{-n}, \cdots, l_{-1}, l_0, l_1, \cdots, l_n, \cdots$, and let E_n be the set for which $l_{n-1} \leqq f < l_n$. We say that f is *summable relative to v and* $\mathfrak{E}$, if there exists a division of $-\infty$ to $+\infty$ for which $\sum_{n=-\infty}^{n=+\infty} l_n v(E_n)$ is absolutely convergent.

If f is measurable and summable relative to v and $\mathfrak{E}$, then we define

$$(L) \int_E f dv = \lim_{d \doteq 0} \Sigma_n \lambda_n v(E_n),$$

where d is the maximum difference $l_n - l_{n-1}$, and $l_{n-1} \leqq \lambda_n < l_n$, which limit can be shown to exist.

These two definitions of integration, the $(Y) \int_E f dv$ and the $(L) \int_E f dv$, are not in general equivalent, the first of the two being the more inclusive in that it may define an integral for functions which are not integrable according to the second definition. For instance, if we suppose that $\mathfrak{P}$ is the linear interval (a, b), and the class $\mathfrak{E}$ is the class of all Borel measurable sets of points, then the (Y) integral method of definition will make all Lebesgue summable functions integrable, while the (L) integral method is restricted to Borel measurable functions. If however we assume that this $\mathfrak{E}$ is extended to include all the Lebesgue measurable sets, i. e., if we add to the class $\mathfrak{E}$ all the sets E for which there exists a Borel measurable set B_1 including it and a Borel measurable set B_2 included within it, for which

$$\text{meas } B_1 = \text{meas } B_2,$$

then the two definitions are equivalent.*

A similar result holds in the more general situation. If we extend the class $\mathfrak{E}$ so as to include all sets E for which there exists an E_1 and an E_2 belonging to $\mathfrak{E}$ such that $E_1 \leqq E \leqq E_2$, and $v(E_1) = v(E_2)$, and call the resulting class *complete as to* v, since no further extension will be possible, we shall have the proposition:

If $\mathfrak{E}$ *is complete as to* v, *then the general Young integral definition is equivalent to the general Lebesgue integral definition, and the integrals of* f *with respect to* v *on any set* E *of* $\mathfrak{E}$ *take the same value.*

There is no difficulty in writing down some of the more important properties of these integrals, either from the corresponding properties of the Stieltjes or the Lebesgue integral. We note only the following:

(1) If $E_1, E_2, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$, and have no common

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* Cf. for instance, de la Vallée Poussin (24), pp. 31–33.

elements, and if $\int_{E_n} f dv$ exists for every n, then if $E = \Sigma_n E_n$,

$$\Sigma_n \int_{E_n} f dv = \int_E f dv.$$

(2) If the f_n are defined on $\mathfrak{P}$ and $\lim_n f_n = f$ on $\mathfrak{P}$, then if the f_n are integrable relative to v on E for every n, and f is summable on E, f will be integrable relative to v and

$$\lim_n \int_E f_n dv = \int_E f dv,$$

if (a) the sequence f_n is monotonic non-decreasing, or (b) $|f_n - f|$ is a bounded function on E and with respect to n.

7. *Special Cases of the Fréchet Integral.*—(Cf. Young (29); Radon (19), pages 1305 ff.) We have already indicated in the preceding paragraph that if we assume $\mathfrak{P}$ to be any linear interval (a, b) and $\mathfrak{E}$ to be the class of all Lebesgue measurable sets of points on (a, b) while $v(E) = \text{meas } E$, the Fréchet integral reduces to the Lebesgue or Young integral, depending upon which manner of definition is followed out. Similarly if $\mathfrak{P}$ is any set of points on (a, b), and $\mathfrak{E}$ is the class of all sets measurable relative to $\mathfrak{P}$, then if we put $v(E) = \overline{\text{meas}}\, E$, we obtain the Pierpont integral for any set E of the class $\mathfrak{E}$.

If we let $\mathfrak{P}$ be the interval (a, b), and $v(E)$ the function of sets defined from the monotonic non-decreasing function $v(x)$ in § 5, and $\mathfrak{E}$ the class of all sets of points measurable relative to $v(x)$, which class will be complete as to v, then we get an extension of the Lebesgue integral to the Stieltjes integral, which, by putting $v(x) = x$, reduces to the ordinary Lebesgue integral, and has properties analogous to those of both the Stieltjes and Lebesgue integrals.*

Young has given an extension of the Stieltjes integral on the basis of monotonic sequences of semi-continuous functions which is equivalent to the one which we have just derived. He defines the value of $\int_a^b f dv$, v being a monotonic non-decreasing function, for functions which are upper or lower semi-continuous but constant on subintervals of the interval (a, b), i. e., what he calls simple u- and l-functions. If $f(x)$

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* Cf. also Daniell, this BULLETIN, vol. 23, pp. 209, 211.

is constant in each of the intervals formed by the points of division $a = x_0, x_1, \cdots, x_n = b$, then his expression can be reduced to

$$\int_a^b f dv = \sum_{i=0}^{n} f(x_i)(v(x_i + 0) - v(x_i - 0)) \\ + \sum_{i=0}^{n-1} f(x_i + 0)(v(x_{i+1} - 0) - v(x_i + 0)),$$

where we assume that $v(x_0 - 0) = v(a - 0) = v(a)$ and $v(x_n + 0) = v(b + 0) = v(b)$.

From this definition of integration for simple upper and lower semi-continuous functions he passes by monotonic sequences of these functions to the integrals of upper and lower semi-continuous functions, and then proceeds in the same way to the general case, i. e., he formulates the following definition:

Form the integrals with respect to $v(x)$ for all upper semi-continuous functions less than the given function, and take the upper bound of these integrals; form the integrals with respect to $v(x)$ of all lower semi-continuous functions greater than the given function and take the lower bound of these integrals; if these two bounds agree the function is said to have an integral with respect to $v(x)$, which is the common value of the bounds.

In showing the equivalence of this definition to the one which we have proposed above, we note first that the value of the integral of simple u- and l-functions as given by Young is the same as that which we should obtain from our definition. The fact that if a function is integrable according to this Young definition, it is also integrable as above, with respect to $v(x)$, is then immediately evident.

To prove the converse, it is sufficient to show that if E is any set measurable relative to $v(x)$, and $f(x)$ has the value unity for x on E and zero elsewhere, then the $(Y) \int_E f dv$ exists, and is equal to $v(E)$. For this purpose we may use the property of $v(E)$ noted previously:

If $v(E)$ exists, then for every $\epsilon > 0$ there exists a closed set E' contained in E such that $v(E') - v(E) < \epsilon$;

and the further fact apparent from the definition of $v(E)$:

If $v(E)$ exists, then for every $\epsilon > 0$, there exists a set of non-overlapping open intervals α_n such that every point of E is interior to some α_n and $\Sigma_n v(\alpha_n) - v(E) < \epsilon$.

Radon has shown how one can define a $v(E)$ with respect to monotonic functions in space of any number of dimensions. So far however none but trivial examples for a general function space have been given. It still remains therefore to give examples of functions $v(E)$ for spaces of infinitely many dimensions, and function spaces, which are not trivial.

It may be of interest to note, finally, that the general Fréchet-Stieltjes integral of a bounded function, when existent on a complete class $\mathfrak{E}$, is expressible as an ordinary Stieltjes integral in the form $\int_l^L y d\mu(y)$, where $l < f < L$ and $\mu(y)$ is the value of $v(E_y)$ where E_y is the set of elements for which $f < y$. This integral in turn is expressible as an ordinary Riemann integral. In a sense then the general Fréchet integral is equivalent to the Stieltjes integral on a linear interval. The extension to the case of an unbounded function is obvious.

IV. The Hellinger Integral.

1. *Definition of the Hellinger Integral and its Relation to the Lebesgue.*—(Cf. Hellinger (11), pages 236 ff.; Hahn (7), pages 170–183; Riesz (20), page 462.) We conclude our discussion of definitions of integration with a brief mention of the Hellinger integral and its generalizations. It is closely related to the Stieltjes integral, as a matter of fact extends in a way the ideas underlying this latter integral, and also had its origin in an attempt to break up into component parts a Stieltjes integral found by Hilbert in his work on quadratic forms in infinitely many variables. It is defined as follows:

Let $v(x)$ be a monotonic non-decreasing continuous function of x in an interval (a, b). Further let $f(x)$ be any continuous function of x, which is constant in the intervals in which $v(x)$ is constant, i. e., if $v(x_2) - v(x_1) = 0$, then $f(x_2) - f(x_1) = 0$. Divide the interval (a, b) into any finite number of intervals by means of the points $a = x_0, x_1, \cdots, x_n = b$, and form the sum

$$\sum_{i=0}^{n-1} \frac{(f(x_{i+1}) - f(x_i))^2}{v(x_{i+1}) - v(x_i)},$$

the quotient being defined to be zero when $v(x_{i+1}) = v(x_i)$. It can then be shown with the aid of the Schwarz inequality that this *sum does not decrease when we subdivide the intervals* (x_i, x_{i+1}). *The least upper bound of this sum for all possible divisions of the interval* (a, b) *is defined to be the Hellinger integral*

$$\int_a^b \frac{(df)^2}{dv}.$$

If the Hellinger integrals

$$\int_a^b \frac{(df_1)^2}{dv} \quad \text{and} \quad \int_a^b \frac{(df_2)^2}{dv}$$

exist, then it can be shown that the Hellinger integral

$$\int_a^b \frac{df_1 df_2}{dv}$$

also exists and that we have

$$\left(\int_a^b \frac{df_1 df_2}{dv}\right)^2 \leqq \int_a^b \frac{(df_1)^2}{dv} \int_a^b \frac{(df_2)^2}{dv},$$

which corresponds to the Schwarz inequality.

Hahn has stated a necessary and sufficient condition under which the Hellinger integral of f with respect to v will exist, and incidentally given the relation between Hellinger and Lebesgue integrals. Suppose we take the inverse $x(v)$ of the monotonic function $v(x)$. This will not have a unique definition at the values of v for which $v(x)$ is constant, but when we substitute it in the continuous function $f(x)$ which is constant where $v(x)$ is constant, we obtain a continuous function $F(v)$ of v defined in the interval $(v(a), v(b))$. Then Hahn's theorem is:

A necessary and sufficient condition for the existence of the Hellinger integral

$$\int_a^b \frac{(df)^2}{dv}$$

is that $F(v)$ *be the indefinite integral of a function* $F'(v)$, *which is summable and of summable square. Moreover we have*

$$\int_a^b \frac{(df)^2}{dv} = \int_{v(a)}^{v(b)} (F'(v))^2 dv.$$

This theorem thus expresses the Hellinger integral in terms of a Lebesgue integral, and a Lebesgue integral in terms of a Hellinger. For the proof of the theorem we refer to Hahn's memoir (7). Riesz previously obtained a similar result for an integral of the form

$$\int_a^b \frac{|df|^p}{(dv)^{p-1}}, \qquad (p > 1),$$

viz., a necessary and sufficient condition that a continuous function $F(x)$ be the indefinite integral of a function $F'(x)$ for which $\int_a^b |F'(x)|^p dx$ with $p > 1$ exists, is that there exist a finite upper bound for the sum

$$\sum_{i=1}^{n} \frac{|F(x_i) - F(x_{i-1})|^p}{(x_i - x_{i-1})^{p-1}}$$

for all possible subdivisions of the interval (a, b) into a finite number of subintervals.

So far the Hellinger integral has shown itself of value in the theory of quadratic forms in infinitely many variables and the related fields, in which it is effective in breaking up such a form into the sum of squares of linear forms in infinitely many variables, thus completing the analogy with the finite case. No simple reduction of this character has been thus far made by using the Lebesgue integral only, even though the Hellinger integral is expressible in terms of the Lebesgue.

2. *Generalizations of the Hellinger Integral.*—(Cf. Radon, pages 1351 ff.; the Moore generalization has been given by Moore in lectures at the University of Chicago, 1915–17.) An extension of the Hellinger integral has been given by Radon. He presupposes a class $\mathfrak{E}$ of sets E of the type discussed in §§ 5 and 6 of Chapter III in connection with the Fréchet general integral, but limited to points in a space of a finite number of dimensions. He assumes further a function $v(E)$ which is monotonic, i. e., $v(E) \geq 0$ for every E of the class $\mathfrak{E}$, and is absolutely additive, i. e., for every denumerable infinity of mutually distinct sets $E_1, E_2, \cdots, E_n, \cdots$ of $\mathfrak{E}$

$$v(\Sigma_n E_n) = \Sigma_n v(E_n).$$

He further assumes that the functions f are defined on the class $\mathfrak{E}$, are absolutely additive and so of bounded variation, and in addition such that if $v(E)=0$ then $f(E)=0$. A set E is then divided into a finite number of subsets $E_1, \cdots, E_n$ of the class $\mathfrak{E}$, and the expression

$$\sum_{i=1}^{n} \frac{|f(E_i)|^p}{(v(E_i))^{p-1}}$$

formed, the quotient being defined to be zero if $v(E_i) = 0$. Then the least upper bound of this expression for all possible finite subdivisions of the set E into sets E_i is defined to be the integral

$$\int_E \frac{|df|^p}{(dv)^{p-1}}.$$

When $p = 2$, this integral reduces to the Hellinger integral if $v(E)$ is formed on the basis of a continuous monotonic non-decreasing function as in § 5 of Chapter III; and to the sum of squares if formed on the basis of a monotonic non-decreasing function, constant except for a denumerably infinite set of points of discontinuity.

Radon proves also a theorem for the space of n dimensions, which corresponds to the Hahn theorem on Hellinger integrals:

A necessary and sufficient condition that the generalized Hellinger integral

$$\int_E \frac{|df|^p}{(dv)^{p-1}}$$

exist, is that there exists a function F defined on the fundamental set $\mathfrak{P}$, such that $f(E) = \int_E F dv$ for every E of $\mathfrak{E}$, and for which the generalized Stieltjes integral $\int_E |F|^p dv$ exists, this last integral being equal to the generalized Hellinger integral of f with respect to v.

The function F which appears in this theorem is of the nature of a derivative of the function $f(E)$ with respect to the monotonic function $v(E)$; in other words, this connects with the idea of the derivatives of functions with respect to monotonic functions or more generally functions of bounded variation. Some results in this direction have recently been obtained by Young,* though there still is undoubtedly a considerable field for investigation.

A generalization of the Hellinger integral in a different direction has been proposed by Moore. He starts from the integral in the form

$$\int_a^b \frac{df_1 df_2}{dv}$$

and observes that it is bilinear in f_1 and f_2, and it is the bilinear aspect of the Hellinger integral which dominates his generalization. The germ of the generalization is contained in the observation that the Hellinger integral

$$\int \frac{df_1 df_2}{dv}$$

can be written as the limit (or the least upper bound if $f_1 = f_2$) of a double sum of the type

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_1(s_i) \omega_\pi(s_i, s_j) f_2(s_j),$$

where $a \leqq s_1 < s_2 \cdots < s_n \leqq b$ is a partition π of the interval (a, b), and $\omega_\pi(s, t)$ is a function of two variables which depends for its value on the values of v and the partition π.

Suppose then that $\mathfrak{P}$ is a class or set of elements p, concerning the character of which nothing is postulated, i. e., they are perfectly general. We denote by σ any finite collection $(p_1, \cdots, p_n)$ of distinct elements of $\mathfrak{P}$. Let there be defined functions $\xi(p)$ which make correspond to every value of p a real or complex number a. We denote by $\bar{\xi}$ the conjugate function, i. e., the function which for every p takes on the values conjugate to those of ξ.

Let $\epsilon(p, q)$ be a function of the two variables p, q, each of which varies over the range $\mathfrak{P}$. We shall assume that $\epsilon(p, q)$ has the following two properties:

(a) $\epsilon(p, q) = \bar{\epsilon}(q, p)$.

(b) For any σ the determinant of the values $\epsilon(p_i, p_j)$, $i, j = 1, \cdots, n$ is positive and not zero.

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* Cf. *Proceedings Lond. Math. Society*, ser. 2, vol. 15 (1916), pp. 35–64. See also de la Vallée Poussin: (24), pp. 67 ff., where the case in which $v(E) = \text{meas}\, E$, i. e., $v(x) = x$ is treated.

Then we define the operation J as follows:

$$J\bar{\xi}\xi = -\lim_{\sigma} \frac{\begin{vmatrix} 0, & \bar{\xi}(p_1), & \cdots, & \bar{\xi}(p_n) \\ \xi(p_1), & \epsilon(p_1, p_1), & \cdots, & \epsilon(p_1, p_n) \\ \cdot & \cdot & \cdot & \cdot \\ \xi(p_n), & \epsilon(p_n, p_1), & \cdots, & \epsilon(p_n, p_n) \end{vmatrix}}{\begin{vmatrix} \epsilon(p_1, p_1), & \cdots, & \epsilon(p_1, p_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \epsilon(p_n, p_1), & \cdots, & \epsilon(p_n, p_n) \end{vmatrix}},$$

in which by

$$\lim_{\sigma} F(\sigma) = a,$$

we mean* that for every $e > 0$, there is a σ_e such that if σ contains σ_e, then $|F(\sigma) - a| \leqq e$.

More recently he has replaced this definition by one in which the least upper bound notion replaces that of limit as to σ. It is as follows:

Form for any σ the least upper bound of the values of

$$\sum_{i=1}^{n} \bar{\xi}(p_i)\eta(p_i) \sum_{i=1}^{n} \bar{\eta}(p_i)\xi(p_i).$$

for all functions η for which we have

$$\sum_{i,j=1}^{n} \bar{\eta}(p_i)\epsilon(p_i, p_j)\eta(p_j) = 1.$$

Then the least upper bound of these least upper bounds for all possible σ is defined to be $J\bar{\xi}\xi$. These two definitions are equivalent, i. e., for every function ξ they yield the same positive finite† or infinite value.

We notice that this operation J is essentially dependent upon the ϵ chosen in any particular situation. If we specify our ϵ we have theoretically determined the operation J, and the functions ξ for which the $J\bar{\xi}\xi$ exists, i. e., is finite. For instance if $\mathfrak{P} = (1, 2, \cdots)$ and $\epsilon(p, q) =$ the Kronecker

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* Cf. *Proceedings Nat. Acad. Sciences*, vol. 1 (1915), p. 630.

† As an instance of a function $\xi(p)$ for which $J\bar{\xi}\xi$ is finite, we might note $\epsilon(p, q)$ for q fixed For we have

$$J\bar{\epsilon}(p, q)\, \epsilon(p, q) = \epsilon(q, q).$$

δ, i. e., zero for $p \neq q$ and unity for $p = q$, then

$$J\bar{\xi}\xi = \sum_{p=1}^{\infty} \bar{\xi}(p)\xi(p)$$

and the functions on which J operates are those for which the sum of squares of absolute values is convergent. More generally if the class $\mathfrak{P}$ is perfectly general, and $\epsilon(p, q)$ is as above, i. e., zero for $p \neq q$ and unity for $p = q$, then the functions ξ for which $J\bar{\xi}\xi$ exists are different from zero only at most at a denumerable set of elements $p_1, \cdots, p_n, \cdots$ and

$$J\bar{\xi}\xi = \Sigma_n \bar{\xi}(p_n)\xi(p_n).$$

The elements $p_1, \cdots, p_n, \cdots$, at which the functions for which $J\bar{\xi}\xi$ exists are different from zero, may differ for different functions. If $\mathfrak{P}$ is the linear interval (a, b) this will then give an operation for the class of functions which are different from zero only at a denumerable set of points, which functions are disregarded in Lebesgue integration.

If $\mathfrak{P}$ is the interval $(a < p \leqq b)$ and

$$\epsilon(s, t) = \begin{matrix} s - a \text{ if } s \leqq t \\ t - a \text{ if } s \geqq t, \end{matrix}$$

then

$$J\bar{\xi}\xi = \int_a^b \frac{d\bar{\xi}d\xi}{dp},$$

i. e., the J-operation reduces to the Hellinger integral and the functions ξ are the continuous functions for which a Hellinger integral exists, i. e., according to the theorem of Hahn (§ 1) the continuous functions which are the indefinite integrals of functions which are summable and have summable squares.

Similarly the Hellinger integral

$$J\bar{\xi}\xi = \int_a^b \frac{d\bar{\xi}d\xi}{d\psi}$$

is obtained if

$$\epsilon(s, t) = \begin{matrix} \psi(s) - \psi(a) \text{ for } s \leqq t \\ \psi(t) - \psi(a) \text{ for } s \geqq t, \end{matrix}$$

where ψ is a properly monotonic increasing function on the interval $a < p \leqq b$.*

Moore has given non-trivial instances of ϵ's in the case in which p and q range over functional spaces. For instance if $\mathfrak{P}$ is the class of all continuous functions $\varphi(x)$ on the interval (a, b) then

$$\epsilon(\varphi_1, \varphi_2) = e^{\int_a^b \varphi_1(x)\varphi_2(x)dx}$$

has the properties required of the ϵ above.† Just what the character of the operation J is in this case does not seem to have been determined.

This general operation had its origin in the desire to obtain for a general range, functional or otherwise, a theory which would be a generalization of Hilbert's theory of biquadratic forms in infinitely many variables, as simplified by Hellinger. In addition to accomplishing this and incidentally throwing light on what is essential in the Hellinger theory, interest undoubtedly attaches to this generalization as being an instance of a bilinear operation in a general situation with instances of a non-trivial character, and for this reason it is bound to be the subject of further consideration and attention.

The treatment of Borel's definition of integration on page 132 ff. is not entirely clear nor accurate. The definition should be interpreted as follows:

$f(x)$ is (B) integrable in case (a) there exists a set of singularities Z denumerable or even of measure zero, such that for every ϵ and for every set of intervals which has total length at most ϵ and is such that each interval of the set contains at least one point of Z, the Riemann integral of $f(x)$ on the complementary set P_ϵ exists, and (b) these Riemann integrals approach a finite limit as ϵ approaches zero. This limit is the (B) integral of $f(x)$ on (a, b).

Proposition (2) is not correct in that the condition given is necessary but not sufficient, an immediate consequence of proposition (3). If, in accordance with a suggestion recently made by Lusin,* we replace in condition (a) above the words "for every ϵ and for every set" by "for every ϵ there exists a set" and call the resulting integral a (BL) integral, then we can state proposition (2) in the following form:

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* Professor Moore informs me that he has recently succeeded in removing from the condition (b) above the hypothesis that the determinant $|\epsilon(p_i, p_j)|$ is not zero. This would allow ψ to be simply monotonic non-decreasing and the interval to be $a \leqq p \leqq b$, which is the case treated by Hellinger.

† Cf. *American Mathematical Monthly*, vol. 24 (1917), pp. 31 and 333.

A necessary and sufficient condition that $f(x)$ be (BL) integrable is that there exist a set of singularities Z, denumerable or of measure zero, such that (a) the Lebesgue integral of f on the set $Z+Z'$ exists, (b) if the (a_n, b_n) are the intervals complementary to $Z + Z'$ then for every n

$$\lim_{a_n' \doteq a_n,\, b_n' \doteq b_n} (R) \int_{a_n'}^{b_n'} f$$

exists and is finite, and this limit is defined to be the integral over (a_n, b_n), (c) if ω_n is the maximum value of

$$\left| \int_{a_n'}^{b_n'} f \right|$$

for $a_n \leqq a_n' \leqq b_n' \leqq b_n$, then $\Sigma_n \omega_n$ is convergent.

If both the (B) and the (BL) integrals exist for a set Z, then the values will be the same. Also if either the (B) or the (BL) integral exist for different sets Z_1 and Z_2 then the resulting integrals are the same.

In the case of (BL) integrability we have the result that every (L) integrable function is (BL) integrable, but not conversely, in as much as (BL) integrable functions may be non-absolutely integrable.

Finally we desire to remark that Borel's first definition of integration† can be interpreted in the sense of (B^*) integrability. The footnote on page 135 should be revised accordingly.

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* Cf. *Annali di Matematica*, ser. 3, vol. 26 (1917), p. 113.

† Cf. *Comptes Rendus*, vol. 150 (1910) pp. 375–7.

GROUPS FORMED BY SPECIAL MATRICES.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society September 4, 1917.)

1. *Introduction.*

IT is well known that every possible substitution on n letters can be represented by a square matrix of order n which has one and only one unity element in each row and in each column while each of its other elements is 0. For brevity we shall call such a matrix in what follows an n-matrix. The determinant of an n-matrix is ± 1, according as the substitution represented by it is positive or negative. In particular, the totality of all the different n-matrices forms a group with respect to multiplication, and this group is simply isomorphic with the symmetric group of degree n. The totality of these matrices whose determinants are equal to unity also forms a group with respect to the same law of composition, which is evidently simply isomorphic with the alternating group of degree n.

Hence the theory of substitution groups is identical with the theory of the multiplication of such special matrices. The main object of the present paper is to exhibit the equivalence of the theory of imprimitive substitution groups and the theory of multiplication of another type of special matrices. We shall first consider the special case where the imprimitive group is of degree $2n$ and has n systems of imprimitivity. Hence each of these systems involves two letters.

The largest possible imprimitive group of this type is clearly of order $2^n \cdot n!$, and is simply isomorphic with the group formed by all the possible different square matrices of order n which have one and only one ± 1 element in each row and in each column, while each of their other elements is 0. Since every possible imprimitive group of degree $2n$ which has n systems of imprimitivity is conjugate with a subgroup of the former of these groups it results that *every possible imprimitive group of degree $2n$ which has n systems of imprimitivity is simply isomorphic with a group formed by square matrices of order n having one and only one ± 1 element in each row and in each column, while all their other elements are* 0.

In fact, the preceding theorem can be stated in a somewhat more general form if it is noted that the letters of such an imprimitive group can be placed in a (1, 1) correspondence with the elements of these matrices if we let the columns correspond to the systems of imprimitivity and the two letters of each system to ± 1. Such a representation may also serve to exhibit in a new light the meaning of the term systems of imprimitivity, and, if it is employed, the words "simply isomorphic" in the preceding theorem may be replaced by the somewhat stronger word "conjugate."

To obtain an elementary matrix notation for every possible imprimitive group it may be noted that every imprimitive group of degree kn which has n systems of imprimitivity is conjugate with a subgroup of the imprimitive group obtained by forming the direct product of n symmetric groups of degree k, and adjoining to this direct product substitutions which permute its systems of intransitivity according to the symmetric group of degree n. Hence it results directly that *every possible imprimitive group of degree kn which has n systems of imprimitivity is conjugate with a group formed by square matrices of order n having one and only one k-matrix element in each row and in each column while all its other elements are* 0. Instead of representing these non-zero elements by k-matrices they may clearly be represented by elements of a group.

2. *Special Imprimitive Groups and Their Invariant Elements.*

The totality of the square matrices of order n which have one and only one kth root of unity element in each row and in each column, while all their other elements are 0, constitute a group G_1 of order $k^n \cdot n!$, which is simply isomorphic with the imprimitive substitution group of degree kn constructed as follows: Form the direct product of n regular cyclic groups of order k, and adjoin to this direct product substitutions which separately permute its systems of intransitivity according to the symmetric group of degree n.

The totality of the *principal diagonal matrices* of G_1, i. e., those matrices whose elements outside of the principal diagonals are entirely composed of zeros, constitutes a subgroup of G_1 which is simply isomorphic with the direct product formed by n cyclic groups of order k. The central of G_1 is of order k, and is composed of the principal diagonal matrices having equal elements, i. e., of the scalar matrices contained in G_1.

A necessary and sufficient condition that an imprimitive group of degree kn which has n systems of imprimitivity can be represented by such square matrices of order n is that the group composed of all the substitutions on the letters of one system of imprimitivity which transform this system into itself is cyclic. In particular, *a necessary and sufficient condition that a regular substitution group can be represented by square matrices of order n having one and only one kth root of unity element in each row and in each column, while each of the other elements is* 0, *is that it contains a cyclic subgroup of index n.* For instance, with respect to the subgroup of order 3, and one of the subgroups of order 2 the symmetric group of order 6 is represented as a regular group by each of the following two sets of six matrices, ω representing an imaginary cube root of unity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega \\ \omega^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega^2 \\ \omega & 0 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Another interesting category of groups is represented by the $[\phi(m)]^n \cdot n!$ matrices, having one and only one element in each row and in each column which is any one of the $\varphi(m)$ positive integers not greater than m and prime to m, while each of the other elements is 0. When $m > 1$ these matrices represent the imprimitive substitution group constructed as follows: Form the direct product of the n regular groups which are separately simply isomorphic with the group formed by the $\varphi(m)$ positive integers less than m and prime to m. To this direct product adjoin substitutions which permute its systems of intransitivity according to the symmetric group of degree n.

The group noted in the preceding paragraph has $\phi(m)$ invariant elements, since the simply isomorphic imprimitive group has this property. These invariant elements are represented by the principal diagonal matrices whose elements are all equal to the same positive integer. This result follows also from the method of transforming n-matrices by each other, which we proceed to explain.

If $r_1, r_2, \cdots, r_n$ and $c_1, c_2, \cdots, c_n$ represent the rows and the columns respectively, taken in order, of an n-matrix, then the substitution represented by this matrix may be denoted as follows:

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ c_{\alpha_1} & c_{\alpha_2} & \cdots & c_{\alpha_n} \end{pmatrix}$$

where $c_{\alpha_1}, c_{\alpha_2}, \cdots, c_{\alpha_n}$ represent the columns containing the unit element in the rows represented by $r_1, r_2, \cdots, r_n$ respectively. Hence any n-matrix can be transformed by any other n-matrix by interchanging the rows and the columns of the former according to the substitution represented by the latter. If any matrix having one and only one non-zero element in each row and in each column is transformed by any other such matrix, the transformed matrix is again of this form and its non-zero elements occur in the same rows and columns irrespective of the value of these non-zero elements. Hence the theorem noted in the preceding paragraph results directly from the fact that identity is the only invariant element of the symmetric group.

While the invariant elements of the imprimitive groups considered above generate invariant subgroups, it should not be assumed that these subgroups give rise to invariant imprimitive subgroups. On the contrary, every invariant imprimitive subgroup of G_1 contains *all* its principal diagonal matrices whose determinants are equal to unity. Hence the principal diagonal matrices of any imprimitive invariant subgroup of G_1 must always include all these matrices whose determinants satisfy the equation $x^d = 1$, where d is a divisor of k. This is equivalent to the following theorem:

If an imprimitive group of degree kn has for its head H, the direct product of n regular cyclic groups of degree k, then every invariant imprimitive subgroup has for its head a subgroup of H which is simply isomorphic with the direct product of $n - 1$ cyclic groups of order k and of some subgroup of such a cyclic group.

By means of this theorem it is not difficult to determine all the invariant subgroups of the groups considered above.

UNIVERSITY OF ILLINOIS.

CELEBRATED PROBLEMS OF GEOMETRY.

Sur les Problèmes célèbres de la Géométrie élémentaire non résolubles avec la Règle et le Compas. Par F. GOMES TEIXEIRA. Coïmbre, Imprimerie de l'Université, 1915.[*] 4to. 132 pp.

THERE have been many historical surveys of the three famous problems of the ancients. One such was Montucla's anonymous work of 1754 on the history of the problem of the squaring of the circle with a supplement concerning the problems of the duplication of the cube and the trisection of an angle.[†] But a more adequate history of the problem of the duplication of the cube was published by Reimer in 1798.[‡] An accurate and still more elaborate presentation which took due account of later research was published about a century later by Ambros Sturm.[§] A. Conti's account of the problems of duplication of the cube and trisection of an angle occupied about 70 pages of the second part of Enriques's Fragen der Elementargeometrie, which appeared in 1907. This same work contained B. Calò's chapter (60 pages) on transcendental problems, especially that of squaring the circle. These chapters underwent some revisions in the new Italian edition: Questioni riguardanti le Matematiche elementari (1914).[||] Prior to Calò's article, one of the best sketches of the history of the problem of squaring the circle was by Rudio, 1892;[¶] Vahlen's discussion[**] (1911) is also valuable; Hobson's most readable history appeared in 1913.

Such are some of the chief historical surveys. The most recent bibliography of the problems is by Professor Guimaraes.* In the writer's opinion the most elementary presentation of the proofs of the impossibility of their solution with ruler and compasses is due to Klein.†

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[*] Also published as an Appendix to Gomes Teixeira's Obras sobre Mathematica, vol 7, 1915, pages 285–412.

[†] Histoire des recherches sur la quadrature du cercle. Paris, MDCCLIV. 47 + 304 pp. + 8 plates. Nouvelle édition revue et corrigée (par S. F. Lacroix). Paris, 1831. 16 + 300 pp. + 4 plates.

[‡] N. T. Reimer, Historia problematis de cubi duplicatione. Gottingæ, MDCCXCVIII. 16 + 222 pp. + 2 plates.

[§] A. Sturm, Das Delische Problem, Linz, 1895–1897. 140 pp.

[||] Vol. 2, Bologna, 1914. Pp. 185–335.

[¶] F. Rudio, Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen uber die Kreismessung. Deutsch herausgegeben und mit einer Uebersicht uber die Geschichte des Problemes von der Quadratur des Zirkels, von den altesten Zeiten bis auf unsere Tage. Leipzig, 1892. History, pp. 1–69.

[**] T. Vahlen, Konstruktionen und Approximationen. Leipzig, 1911. Pp. 175 ff. and 306 ff.

Since Montucla's work is very scarce, those unacquainted with German or Italian who wished to learn the main facts in such surveys as the ones to which I have referred, have had, till recently, considerable difficulty in satisfying their desires. We have now, however, the very interesting and excellent volume under review, of Professor Gomes Teixeira, Rector of the University of Porto. His power of lucid exposition and his scholarly style are probably familiar to many Americans through the two-volume Traité des Courbes spéciales remarquables planes et gauches of 1908–09.‡

Nearly the whole of the volume on "problèmes célèbres" is given over to a consideration of the three famous problems of the ancients. Chapter I (pages 5–46) is entitled: "Sur le problème des moyennes proportionnelles. Duplication du cube;" Chapter II (pages 47–82): "Sur la division de l'angle;" Chapter III (pages 83–104): "Sur la quadrature du cercle;" and the last chapter: "Sur l'impossibilité de la résolution par la règle et le compas des problèmes considérés précédemment." There are many references to the author's treatise on curves and it is especially in this connection that new features are introduced.

For example, in the first chapter we have: the curve of Archytas—a skew curve; the kampyle of Eudoxus, the simple folium in the method of Villapandus, and the conchoid of Nicomedes—quartics; the hyperbola mesolabica of Viviani, the circular unicursals in the solutions of Plato and Diocles, and the right strophoid in connection with Huygens's solution—cubics; the method of Menæchmus—by conics; and so on. The chapter contains also solutions by Hero of Alexandria, Philo of Byzantium, Apollonius, Eratosthenes, Viète, Descartes, Fermat, Newton, Clairaut, and Montucci.

The second chapter sets forth the methods of Hippias, Archimedes, Nicomedes, Pappus, Etienne Pascal (with his

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* R. Guimaraes, "Algunas consideraciones sobre tres problemas célebre de geometría elemental," *Revista de la Sociedad matemática Española*, año 6, Enero-Abril, 1917, pp. 18–27, 74–94.

† F. Klein, Vortrage über ausgewählte Fragen der Elementar-Geometrie. Leipzig, 1895. English translation by Beman and Smith, Boston, 1897.

‡ Pages 1–284 of Gomes Teixeira's Obras sobre Mathematica, vol. 7 (1915), contain five chapters supplementary to this work.

limaçon), Descartes and Fermat, Kinner, Ceva, Maclaurin, Delanges, Chasles, Lucas, Catalan, Longchamps, and Kempe, with many interesting connections and generalizations. It is shown that the solution of the following problem of Archimedes reduces to that of the trisection of an angle: "To cut a sphere by a plane so that the volumes of the segments are to one another in a given ratio." At this point it would have been interesting to have added a reference to Brocard's pamphlet, Mémoire sur divers problèmes de géométrie dont la solution dépend de la trisection de l'angle (Algiers, 1912).

Viète and Descartes stated that the solution of any problem depending on an equation of the third degree could be reduced to the solution of a problem of finding two mean proportionals, or to that of the trisection of an angle. With a proof of this theorem, and some general remarks, the chapter concludes.

"Given a fixed conic (except circle and line pair) in the plane of construction every problem of the third order can be carried through with ruler and compasses." Professor Gomes Teixeira has apparently followed Vahlen in crediting this theorem to "S. Smith" (1868). To the Englishman, "H. J. S. Smith" seems more natural.

"All problems of the third order can be carried through with ruler alone if a complete fixed curve of the third order is given; for metrical problems a square (or rectangle) must also be given." This theorem was shown by London in 1896. Its statement on page 82 needs to be revised. Compare Conti's article. In his Arithmetica Universalis Newton solved cubic and biquadratic equations by means of the conchoid and ruler and compasses.

In the third chapter there are a number of unusual expressions for π, and its powers, taken from the writings of Wallis, Euler, and Cauchy. Due credit is given to Chinese discoveries in accordance with Mikami's History. On page 88, line 14, for Chang Hing read Chang Hêng.*

The derivation of the results leading to the fundamental theorem by means of which it is shown that the problem of the duplication of the cube is impossible is based mainly on the discussion in Petersen's Theory of Equations. The work concludes with Klein's proof of the impossibility of the problem of squaring the circle with ruler and compasses.

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* Some fairly obvious misprints occur at the following places: page 25, line 4 from bottom; page 26, line 17; page 56, line 12; page 68, line 6 from bottom, page 82, line 6; page 103, last line; page 105, line 6 from bottom; page 122, line 25.

On page 122 occurs the sentence: "La divisionde la circonference en 3 et 5 parties égales a été considérée dans les Elements d'Euclide." Quite true; but why not have written "3, 4, 5, 6 et 15 parties égales"? The statement that the first geometric construction of the regular polygon of 17 sides was found by Erchinger needs revision. Gauss reported Erchinger's paper in 1825 and pointed out that its merit was not so much in the construction as in the synthetic proof of its correctness. Indeed Gauss himself refers to two earlier constructions by Paucker.* At least two more were published before 1825; one by John Lowry in 1819† and the other by Samuel Jones in 1820.‡

We heartily recommend Professor Gomes Teixeira's book for every mathematical library, as no other publication of the kind can take its place. The little book is characterized by marked individuality. When a new edition is called for we hope that the author may be moved to add another chapter on still more of the many famous problems of the category he has been considering. For example, an adequate history of the following century-old problem has not yet been published: "Given the length of the bisectors of the angles of a triangle between the vertices and the opposite sides to construct the triangle." In 1911, Professor R. P. Baker published a hundred-page doctor's dissertation on this problem.

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* (*a*) "Geometrische Verzeichnung des regelmassigen 17-Ecks und 257-Ecks in d. Kreis," *Jahresverhandl. d. kurlandische Gesellschaft fur Literatur und Kunst*, Mitau, Band 2, 1822. (*b*) Die ebene Geometrie der geraden Linie und des Kreises, Konigsberg, 1823, p. 187.

† *The Mathematical Repository*, new series, vol. 4 (1819), p. 160.

‡ The paper dated "Dublin, 17th October, 1819" and read January 24, 1820, was published in *Transactions of the Irish Academy*, vol. 13 (1818), pp. 175–187.

NOTES.

AT the annual meeting of the London mathematical society, held November 1, the following papers were read: By J. H. GRACE, "Tetrahedra in relation to spheres and quadrics"; by M. J. M. HILL, "The continuation of the hypergeometric series"; by W. H. YOUNG, "Restricted Fourier series and the convergence of power series"; by E. B. STOUFFER, "Invariants and covariants of linear homogeneous differential equations"; by H. W. TURNBULL, "The simultaneous system of two quaternary quadratic forms." Professor H. M. MACDONALD was elected president, Professors H. HILTON and E. W. HOBSON and Sir J. LARMOR vice-presidents, and Dr. T. J. I'A. BROMWICH and MR. G. H. HARDY secretaries.

AT the meeting of the Edinburgh mathematical society on November 9, the following papers were read: By W. P. MILNE, "The apolar locus of two tetrads of points on a conic"; by F. G. W. BROWN, "The Brocard and Tucker circles of a cyclic quadrilateral."

PROFESSOR G. A. GIBSON, of the University of Glasgow, has been elected one of the vice-presidents and Professor E. T. WHITTAKER, of the University of Edinburgh, one of the secretaries of the Royal Society of Edinburgh.

THE December number (volume 19, number 2) of the *Annals of Mathematics* contains the following papers: "Factorization of analytic functions of several variables," by W. F. OSGOOD; "An application of Fourier's series to probability," by D. F. BARROW; "Conjugate systems of curves both of whose Laplace transforms are lines of curvature," by J. M. STETSON; "A theorem on lattice-points," by A. J. KEMPNER; "On continuous representations of a square upon itself," by H. L. SMITH; "Roots and singular points of semi-analytic functions," by DUNHAM JACKSON; "Some properties of polynomial curves," by FRANK IRWIN and H. N. WRIGHT.

ON account of conditions due to the war, no Benjamin Peirce instructors will be appointed at Harvard this year.

THE following university and college teachers of mathematics have recently entered the national military service:

PROFESSOR L. K. ADKINS, of the Wisconsin State Normal School at La Crosse, has been appointed first lieutenant in the regular army and ordered to France. Associate professor C. A. EPPERSON, of the Normal School at Kirksville, Mo., is in military training at Fort Sheridan. Mr. M. HEDLUND, of Beloit College, has been appointed second lieutenant in the Adjutant General's office. Mr. H. H. PRIDE, of New York University, has been appointed second lieutenant in the National Army. Assistant professor P. L. THORNE, of New York University, is in military training at Plattsburg. Assistant professor C. H. YEATON, of Northwestern University, has joined the Signal Corps.

MR. W. D. LAMBERT, of the Coast and Geodetic Survey, has been appointed first lieutenant in the army.

PROFESSOR A. O. LEUSCHNER, of the University of California, has been delegated to administer the details of instruction in the navigation schools of the United States Shipping Board on the Pacific coast, and in particular to provide qualified instructors.

PROFESSOR H. E. HAWKES, of the department of mathematics, Columbia University, has been made acting dean of Columbia College during the absence of Dean Keppel on government service.

AT Western Reserve University, MR. D. R. BELCHER, of Columbia University, has been appointed instructor in mathematics in Adelbert College. At the College for Women, Professor ANNA H. PALMIÉ has been granted leave of absence, Mrs. W. E. BECKWITH has been promoted to an assistant professorship, and Dr. MARY F. CURTIS has been appointed instructor in mathematics.

DR. J. E. MCATEE has been appointed associate professor of mathematics at William Jewell College.

AT the University of Maine, assistant professor T. L. HAMLIN has resigned to accept a position in the department of mathematics at Union College and associate professor H. R.

WILLARD has been appointed statistician under Mr. Hoover at Washington.

ASSOCIATE professor E. H. JONES, of the Southern Methodist University, Dallas, Texas, has been promoted to a full professorship of mathematics.

DR. G. H. LIGHT, of the University of Colorado, has been promoted to an assistant professorship of mathematics.

AT the University of Iowa, associate professor R. P. BAKER has been made acting head of the department of mathematics. Mr. R. E. GLEASON and Mr. F. M. WEIDA have been appointed instructors in mathematics.

AT Louisiana State University, assistant professor S. T. SANDERS has been made head of the department of mathematics, and Dr. I. C. NICHOLS has been appointed associate professor.

DR. GOLDIE P. HORTON has been made instructor in mathematics at the University of Texas.

MR. C. H. CLEVENGER, of the University of Minnesota, has resigned to undertake industrial research at the University of Illinois.

MR. J. B. ROSENBACH has been appointed instructor in mathematics at the University of New Mexico.

MRS. M. LOGSDON, of Hastings College, has been appointed instructor in mathematics at Northwestern University.

MR. O. W. ALBERT has been appointed instructor in mathematics at Grinnell College.

MR. L. H. RICE, of Syracuse University, has been appointed instructor in mathematics at Tufts College.

MISS MARION E. STARK, of Brown University, has been appointed professor of mathematics at Meredith College, Raleigh, N. C.

MR. VERN JAMES, of Indiana University, has been appointed instructor in mathematics at the Carnegie Institute of Technology.

MR. C. T. LEVY, of the University of Oklahoma, died July 23, 1917.

PROFESSOR S. F. NORRIS, of Baltimore City College, died September 4, 1917.

PROFESSOR J. F. MILLIS, of the Parker School, Chicago, died October 25, 1917, at the age of forty-two years. He was joint author of the Stone-Millis series of elementary mathematical text-books.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMOROSO (L.) e BOMPIANI (E.). Esercizi di geometria analitica e proiettiva, con prefazione del G. Castelnuovo. Pavia, Mattei, 1917. 8vo. 12 + 475 pp. L. 16.00

BOMPIANI (E.). See AMOROSO (L.).

CASTELNUOVO (G.). See AMOROSO (L.).

HERMITE (C.). Oeuvres de Charles Hermite publiées sous les auspices de l'Académie des Sciences par E. Picard. Tome IV et dernier. Paris, Gauthier-Villars, 1917. 8vo. 6 + 596 pp. Fr. 25.00

MERZ (K.). Zur Erkenntnistheorie uber Raum und Zahl, aus Historischem der Steinerschen Fläche. Croire, Librairie Schuler, 1917. 8vo. 48 pp. Fr. 1.00

D'OCAGNE (M.). Cours de géométrie pure et appliquée de l'Ecole Polytechnique. Tome I: Transformations géométriques. Perspective. Géométrie infinitésimale. Géométrie réglée. Géométrie cinématique. Paris, Gauthier-Villars, 1917. 8vo. 12 + 375 pp. Fr 16.00

PHILLIPS (H. B.). Differential and integral calculus. New York, Wiley, 1916–1917. 5 + 162 + 5 + 194 pp. Cloth. $2.00

——. Integral calculus. New York, Wiley, 1917. 5 + 194 pp. Cloth. $1.25

PICARD (E.). See HERMITE (C.).

PROMPT (–.). Recherches analytiques sur les carrés magiques. Paris, Gauthier-Villars, 1917. 8vo. 34 pp. Fr. 2 20

SKINNER (E. B.). College algebra. New York, Macmillan, 1917. 12mo. 8 + 263 pp. Cloth. $1.50

II. ELEMENTARY MATHEMATICS.

BAUER (G. N.) and BROOKE (W. E.). Plane and spherical trigonometry Second revised edition. Boston, Heath, 1917. 12 + 174 pp.

——. Logarithmic and trigonometric tables. Boston, Heath, 1917. 4 + 139 pp.

BRESLICH (E. R.). Logarithmic and trigonometric tables and mathematical formulas. Chicago, University of Chicago Press, 1917. 17 + 118 pp. Cloth. $0.75

———. Third-year mathematics for secondary schools. With tables to five places. Chicago, University of Chicago Press, 1917. 18 + 369 pp. $1.50. Without tables, $1.00

Brooke (W. E). See Bauer (G. N.).

Brown (J. C.). See Wentworth (G.).

Cadorius (I.), Fisker (A.), Herlev (V.) og Olsen (L P.). Regnebogen. Lærerens Bog 1–2 og Elevens Bog 1–3. København, Gyldendalske Boghandel, Nordisk Forlag, 1916.

Counts (G. S). Arithmetic tests and studies in the psychology of arithmetic. Chicago, University of Chicago Press, 1917. 4 + 127 pp. Paper. $0.75

Fisker (A.). See Cadorius (I.).

Gentleman (F. W.). See Vosburgh (W. L.).

Herlev (V.). See Cadorius (I.).

Houel (J.) Tables de logarithmes à cinq décimales pour les nombres et les lignes trigonométriques. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1917. 8vo. Cartonné. Fr. 3 85

Mathematical Association. Elementary mathematics in girls' schools. Report of the girls' schools committee of the Mathematical Association, 1916. London, Bell, 1916. 8vo. 26 pp. 1s.

Milne (R M). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, February-July, 1917. London, Macmillan, 1917. Cr. 8vo. 1s.

Monroe (W. S.). Development of arithmetic as a school subject. Washington, Government Printing Office, 1917. 170 pp. Paper. $0 20

Monteil (P. L.). Mesure de la longueur de la circonference par la valeur $\pi R = R\sqrt{3} + R\sqrt{2}$. Paris, Dunod et Pinat, 1917. 10 + 57 pp.

Olsen (L. P.). See Cadorius (I.).

Smith (A.). The new Barnes problem books, seventh and eighth years, first and second halves. New York, A. S. Barnes, 1917. 70 pp. each. Paper. $0.10 + 0.10

Smith (D. E.). See Wentworth (G.).

Vosburgh (W. L.) and Gentleman (F. W). Junior high school mathematics. New York, Macmillan, 1917. 12mo. 7 + 146 pp. $0.75

Wentworth (G.), Smith (D. E.) and Brown (J. C.). Junior high school mathematics. Books 1 and 2. Boston, Ginn, 1917. Each $0.76

III. APPLIED MATHEMATICS.

Benoit (J. R.) et Guillaume (C. E.). La mesure rapide des bases géodésiques. 5e édition. Paris, Gauthier-Villars, 1917. 8vo 285 pp. Fr. 7 70

Bigourdan (G.). Les méthodes d'examen des lunettes et des télescopes. Paris, Gauthier-Villars, 1915. 8vo. 216 pp. Fr. 6.00

Connaissance des temps ou des mouvements célestes pour le méridien de Paris à l'usage des astronomes et des navigateurs pour l'an 1919, publiée par le Bureau des Longitudes. Paris, Gauthier-Villars, 1917. 8vo. 30 + 736 pp. Fr. 4.40

Doberck (W.). Hygrometric tables for use with rotating dry and wet bulb thermometers. London, Williams and Norgate, 1917. 2s. 6d.

DOOLEY (W. H.). Vocational mathematics. Boston, Heath, 1917. 8 + 341 pp. Cloth.

———. Vocational mathematics for girls. Boston, Heath, 1917. 6 + 369 pp. Cloth.

ECHEGARAY (J.). Conferencias sobre fisica matemática. Madrid, Estabecimiento Tipográfico Editorial. Curso de 1910–11: Teoria de los torbellinos. 1911. 394 pp. Curso de 1911–12: Teorias diversas. 1912. 582 pp. Curso de 1912–13: Ecuaciones de la mecánica. 1913. 531 pp. Curso de 1913–14: Teoria de los torbellinos (segunda parte). 1914. 550 pp

ESPIN (T. E.). See WEBB (T. W.).

GUILLAUME (C. E.). See BENOIT (J. R.).

HOLMES (H. W.) and others. A descriptive bibliography of measurement in elementary subjects. (Harvard Bulletins in Education, Vol. V.) Cambridge, Harvard University Press, 1917. 8vo. 7 + 46 pp.

JEKOWSKY (B.). Exposé succinct, suivi d'un exemple numérique, d'une méthode nouvelle pour la détermination d'une orbite parabolique. Paris, Gauthier-Villars, 1917. 8vo. 20 pp. Fr. 2 20

MCNICOL (D.). La télégraphie en Amérique, traduit de l'anglais par E. Picault et G. Viard. Paris, Gauthier-Villars, 1917. 8vo. 6 + 278 pp. Fr. 12.10

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LIBRARY

THE ELEVENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

The eleventh regular meeting of the Southwestern Section of the Society was held on December 1, at Norman, Oklahoma, in the administration building of the University of Oklahoma, Professor S. W. Reaves presiding. The sessions opened at 10.30 A.M. and 2.30 P.M. respectively. Attending members were entertained at a smoker on the previous evening and at lunch on the day of the meeting. There were present fourteen persons, including the following eleven members of the Society:

Professor Nathan Altshiller, Professor C. H. Ashton, Professor Henry Blumberg, Professor E. W. Davis, Professor E. P. R. Duval, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. B. B. Libby, Mr. E. D. Meacham, Professor S. W. Reaves, and Professor W. H. Roever.

It was decided that the next meeting of the Section should be held on November 30, 1918, at Columbia, Missouri, and that the program committee should be Professors E. R. Hedrick (chairman), W. C. Brenke, O. D. Kellogg (secretary).

The following papers were presented at this meeting:

(1) Professor A. M. HARDING: "Rational plane anharmonic cubics."

(2) Professor NATHAN ALTSHILLER: "On the I-centers of a triangle."

(3) Professor R. D. CARMICHAEL: "Fermat numbers $F_n = 2^{2^n} + 1$."

(4) Dr. P. R. RIDER: "An intrinsic equation solution of a problem of Euler."

(5) Professor O. D. KELLOGG: "Interpolation properties of solutions of certain differential equations."

(6) Professor S. LEFSCHETZ: "On multiple integrals belonging to an algebraic variety."

(7) Professor NATHAN ALTSHILLER: "On the Teixeira construction of the unicursal cubic."

(8) Professor HENRY BLUMBERG: "Non-measurable functions connected with functional equations."

(9) Professor HENRY BLUMBERG: "A theorem on semicontinuous functions."

(10) Professor J. E. MCATEE: "Polynomial modular invariants of a binary quadratic."

(11) Professors E. R. HEDRICK and LOUIS INGOLD: "A generalization of Bessel's inequality and related formulas."

(12) Professor W. H. ROEVER: "Geometric explanation of a certain optical phenomenon."

Professor McAtee's paper was communicated to the Society through Professor Fleet. In the absence of the authors the papers of Professors Harding, Carmichael, Lefschetz, and McAtee were read by title, and Dr. Rider's paper was presented by Professor Roever

Abstracts of the papers follow in the order of their titles above.

1. In a recent paper (*Giornale de Matematiche*, volume 54, 1916) Professor Harding has studied certain projective properties of anharmonic curves and has found the coordinates of the invariant triangle in terms of the invariants of a certain differential equation. The present paper is limited to a discussion of the properties of rational anharmonic cubic curves. It is shown that each of these curves has one cusp and one point of inflection, and that the invariant triangle is formed by the cuspidal tangent, the inflectional tangent, and the line joining the cusp to the point of inflection.

2. Professor Altshiller's paper will soon appear in the *American Mathematical Monthly*.

3. In this paper, Professor Carmichael gives elementary proofs of all the essential known facts about the Fermat numbers $F_n = 2^{2^n} + 1$ and a derivation of some new results of minor importance.

4. The following problem was proposed and solved by Euler: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. Making use of a method developed by Radon for minimizing the integral $\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds$, Dr. Rider gives an intrinsic equation solution of the problem which is much shorter than the solution ordinarily given.

5. Continuing the work reported on at the summer meeting, (see BULLETIN, November, 1917, page 62) Professor Kellogg

extends to the case of the general self-adjoint homogeneous boundary conditions the results of Liouville on the possession of the interpolation property (D) by the solutions of linear differential equations of second order. The results will appear in the *American Journal of Mathematics*.

6. Let $F(x, y, z, t) = 0$ be the equation of an algebraic three-dimensional variety in S_4 and consider the integrals

$$J = \int\int\int \frac{P(x, y, z, t)}{F_t'} dxdydz, \quad J = \int\int \frac{P(x, y, z, t)}{F_t'} dxdy$$

belonging to F and to its hyperplane sections $z = C$, or H_z. The fundamental theorem proved by Professor Lefschetz in his paper is this: The periods of J_z are algebraic functions of z. This makes it possible to replace J as far as its periods are concerned by a certain abelian integral. The number of these periods is found to be $R_3' = I + 2R_2 - 3R_1 - 4$ (I invariant of Zeuthen-Segre, R_i i-dimensional connectivity). By comparing with a result of Alexander it is seen that $R_3' = R_3 - R_1$, which agrees with the existence of R_1 tridimensional cycles relative to which the periods of J are zero. The definition of triple integrals of the second kind is about the same as Picard's for algebraic surfaces. Their reduction and enumeration is closely related to a class of double integrals of the type

$$\int\int Udydz + Vdzdy + Wdxdy; \qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0,$$

and to an invariant number λ analogous to Picard's number ρ. The number of integrals of the second kind turns out to be $\rho_0 = R_3' - \lambda$. Extensions to higher varieties are also considered. A summary of these results appeared in the *Comptes Rendus* of May 29, 1917.

7. Consider a point O, a line s, a conic C, and two points D and A on s and C respectively. A variable line through A meets s in P and C again in B. Let $M \equiv (OP, DB)$. The locus of the point M is, in general, a unicursal cubic having O for its double point, passing through D and through the points common to s and C. Conversely, with an arbitrarily chosen straight line an infinite number of conics may be associated in

order to generate a given unicursal cubic by the above method, provided the straight line does not pass through the double point of the cubic and is not an inflectional tangent. Professor Altshiller proves synthetically and discusses this proposition, which is a generalization of a theorem due to F. Gomes Teixeira (*Nouvelles Annales de Mathématiques*, August, 1917, pages 281–284).

8. At the 1916 meeting of the Southwestern Section, Professor Blumberg communicated—among other things—the fact that discontinuous solutions of the functional equation $f(x+y) = f(x) + f(y)$ are necessarily non-measurable. This holds essentially also for the equation $f(xy) = f(x)f(y)$. These examples suggest the generalized relation

$$f[\varphi(x, y)] = \varphi[f(x), f(y)],$$

where φ is to be regarded as a fixed function of two variables and f as the variable solution of the functional equation. Under suitable assumptions for φ, a large class of functional equations is obtained whose discontinuous solutions are necessarily non-measurable.

9. The theorem of Professor Blumberg's second paper is as follows: Let φ_{ab} be a monotone decreasing, real interval-function; i. e., φ_{ab} is a real number for every (closed) interval (a, b), and, in addition, $\varphi_{ab} \leqq \varphi_{cd}$ for (a, b) within (c, d). Moreover, let φ_{ab} have a finite lower bound. Let $\varphi(x)$ be the point function associated with φ_{ab}; i. e., $\varphi(x)$ is the greatest lower bound of all φ_{ab} such that x is in the interior of (a, b); let $\varphi^{(r)}(x)$ —the symbol (r) designating "right"—be the greatest lower bound of all φ_{xb}, and $\varphi^{(l)}(x)$, the greatest lower bound of all φ_{ax}. Then each of these three functions of x is an upper semicontinuous function—this part of the theorem is in essence not new—and $\varphi(x) = \varphi^{(r)}(x) = \varphi^{(l)}(x)$ except at most in a countable set. It follows that the saltus function $s(x)$ is, except at most in a countable set, indentical with $s^{(r)}(x)$ and $s^{(l)}(x)$— whose meaning is self-explanatory—and similarly for the f-saltus, d-saltus, etc. (Cf. author's paper, "Certain general properties of functions," *Annals of Mathematics*, March, 1917.) As a very special case, it follows that a function continuous on the right (left) except at most at the points of a countable set is everywhere continuous except possibly at the points of a countable set.

10. Professor Dickson has found certain polynomial modular invariants and properties of such invariants for the case of a binary n-ic and a prime modulus P. In this paper Professor McAtee generalizes these invariants for the case of a binary quadratic and modulus a power of a prime, P^λ. Then these invariants are specialized for the case $P = \lambda = 2$ and a fundamental system is exhibited modulo 4.

11. With a view toward applications to the expansion of functions in terms of given functions, Professors Hedrick and Ingold have been led to study the properties of a general linear distributive operation $L[f(x), \varphi(x)]$ on a pair of functions $f(x)$ and $\varphi(x)$. In this paper, it is shown that such an operation leads at once to a general formula of which Bessel's inequality is a special case. Other related formulas are also generalized.

A similar operator, defined only for the product $f(x) \cdot \varphi(x)$ has been studied by Moore. (See BULLETIN, volume 18, pages 334–362.)

12. Professor Roever gives a geometric explanation of elliptical light curves seen in a highly scratched plate illuminated by a point source.

O. D. KELLOGG,
Secretary of the Section.

NOTE ON CONJUGATE NETS WITH EQUAL POINT INVARIANTS.

BY DR. G. M. GREEN.

(Read before the American Mathematical Society, September 4, 1917.)

IN my second memoir on conjugate nets on a curved surface,[*] I gave a new characterization of conjugate nets with equal Laplace-Darboux invariants. The theorem as there stated, however, is not quite complete, and it is the purpose of this note to supply the necessary refinement, as well as to generalize the theorem and put it into relation with another, which is involved in a different characterization of conjugate nets with equal invariants.

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[*] *Amer. Journal of Mathematics*, vol. 38 (1916), pp. 287–324. See in particular p. 313.

Using the notation of the paper just cited, let

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3, 4)$$

be the point equations, in homogeneous coordinates, of a surface S referred to a conjugate net. The four functions y are a fundamental system of solutions of the completely integrable system of partial differential equations

$$(1) \qquad \begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \\ y_{uv} &= b'y_u + c'y_v + d'y. \end{aligned}$$

The minus first and first Laplace transforms of the point y of the surface are respectively

$$\rho = y_u - b'y, \quad \sigma = y_v - a'y.$$

The line joining these two points we shall call with Wilczynski the ray of the point y, and the congruence of rays the ray congruence. The developables of the ray congruence correspond to a net of curves on S, which we shall call the ray curves, and whose differential equation is*

$$(2) \qquad aHdu^2 - \mathfrak{D}dudv - Kdv^2 = 0,$$

where

$$H = d' + b'c' - b_u', \quad K = d' + b'c' - c_v'$$

are the Laplace-Darboux invariants, and

$$\mathfrak{D} = d + ab'^2 - c'^2 + ab_v' - c_u' + b'c + bc'.$$

Two ray curves pass through each point y of the surface, and we shall call their tangents at the point y the ray tangents of y. The ray tangents are the lines which join y to the points $y_u + (dv/du)_1 y_v$, $y_u + (dv/du)_2 y_v$, where $(dv/du)_1$ and $(dv/du)_2$ are the roots of equation (2) considered as a quadratic in dv/du.

The focal points of the ray $\rho\sigma$ are given by the formulas†

$$R = \rho + r_1\sigma, \quad S = \rho + r_2\sigma,$$

where r_1 and r_2 are the roots of the quadratic

$$(3) \qquad Hr^2 + \mathfrak{D}r - aK = 0.$$

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* G. M. Green, loc. cit., p. 309.

† Loc. cit., p. 309.

A comparison of equations (2) and (3) will show that *the ray tangents meet the ray in the focal points of the ray if and only if the conjugate net has equal Laplace-Darboux invariants, provided, however, that* $\mathfrak{D}$ *is not zero.*[*] If $\mathfrak{D}$ is zero, the quadratics (2) and (3) would coincide not only if $H = K$, but also if $H = -K$. If, however, $H = -K$, and $\mathfrak{D}$ is zero, equation (2) becomes

$$H(adu^2 + dv^2) = 0.$$

If, then, H is different from zero (if it were zero, we should have again $H = K$) the differential equation of the ray curves becomes

$$adu^2 + dv^2 = 0,$$

which is the differential equation of the asymptotics. In other words, *if the ray curves coincide with the asymptotic curves, then the ray tangents will meet the ray in the focal points of the ray.*

However, the geometric facts enable us to state the entire situation more sharply. An analytic proof of these facts—we shall presently supply a geometric proof—will show that in the first case, i. e., $H = K$, the tangent to one of the ray curves through y meets the ray in the focal point which corresponds to the other ray curve, whereas in the second case, i. e., $\mathfrak{D} = 0$ and $H = -K$, the tangent to a ray curve meets the ray in the focal point belonging to that ray curve. So we may state our characterization of conjugate nets with equal invariants as follows:

A necessary and sufficient condition that a conjugate net have equal point invariants is that for every point of the surface the tangents to the two ray curves at the point meet the corresponding ray in the focal points of the ray, the tangent to either ray curve meeting the ray in the focal point which corresponds to the other ray curve. If each ray tangent meets the ray in the focal point which corresponds to it, the ray curves coincide with the asymptotics of the surface. In this case $H = -K$ *and* $\mathfrak{D} = 0$. *If* $H = K = \mathfrak{D} = 0$, *the ray curves are indeterminate.*

In this connection it will be well to recall Wilczynski's theorem,[†] that *a necessary and sufficient condition that a con-*

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[*] Loc. cit., p. 313, except for the proviso at the end of the present statement.

[†] E. J. Wilczynski, "The general theory of congruences," *Transactions Amer. Math. Society*, vol. 16 (1915), pp. 311–327.

jugate net have equal point invariants is that its ray curves form a conjugate net. Using this theorem, we may prove the previous one on the basis of the following very general theorem:

Let t_1 and t_2 be the ray tangents of a point of the surface, and Φ_1, Φ_2 the corresponding focal points, respectively, of the ray. Then the ray is met in Φ_1 by the tangent to the surface which is conjugate to t_1, and in Φ_2 by the tangent conjugate to t_2.

We shall give a geometric proof of this theorem; the proof will show that a perfectly good theorem may sometimes be obtained by merely dualizing a trivial statement. Let us regard the surface S as the locus of points y, and the envelope of the corresponding tangent planes Π. By polar reciprocation, S goes into a surface S_1, the point y becoming the tangent plane Π_1, and the plane Π becoming the point of contact y_1 on S_1. The points ρ and σ become the osculating planes of the parametric curves $u = \text{const.}$, $v = \text{const.}$, respectively, at the point y_1 on S_1. These osculating planes meet in a line, which Wilczynski has called the axis of the point y_1. The totality of axes constitute the axis congruence, and the curves on S_1 which correspond to the developables of this congruence are called the axis curves. The points of a curve C on S, i. e., a one-parameter family of points, correspond to a one-parameter family of planes tangent to S_1. The points of contact of these planes constitute a curve C_1 on S_1, which we may consider as the curve corresponding to C. But the tangent to C at y does not go over into the tangent to C_1 at y_1, but into the conjugate direction. The focal points of the ray $\rho\sigma$ correspond to the focal planes[*] of the axis congruence of S_1. An axis tangent on S_1, i. e., a tangent to an axis curve, corresponds to the direction on S which is conjugate to a ray tangent on S. If we recall that a focal plane of a line of a congruence is tangent to a developable of the congruence all along a line of the congruence, and if we observe that the developables of the axis congruence actually cut the surface S_1 in the axis curves, then it becomes evident that *the axis tangents lie in the corresponding focal planes.* If we dualize this situation on S_1, we obtain the situation on S which is described in the theorem which we set out to prove.

Having, then, shown that the direction conjugate to one ray tangent meets the ray in that focal point of the ray which corresponds to the other ray tangent, and recalling Wilczynski's theorem, we are led at once to the final form in which we stated the geometric characterization of conjugate nets with equal point invariants. This characterization may of course be established analytically, and independently of Wilczynski's theorem, so that, on the basis of our general theorem, Wilczynski's characterization may then be obtained from ours.

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[*] The focal planes of an axis are the planes tangent, along the axis, to the two developables of the axis congruence to which that axis belongs.

HARVARD UNIVERSITY,
August 22, 1917.

ON THE DIFFERENTIABILITY OF ASYMPTOTIC SERIES.

BY DR. J. F. RITT.

(Read before the American Mathematical Society, October 27, 1917.)

THE question of the differentiability of asymptotic series seems not to have received adequate treatment.* Writers on the theory of asymptotic convergence content themselves always with stating that if $P(x)$ has the asymptotic representation

$$P(x) \sim a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

it may not have a derivative at all, and that even if a derivative does exist, the derivative may not admit of asymptotic development.

A failure to distinguish between the real and complex domains, in this connection, is responsible for a serious lacuna, which it is the purpose of this note to fill.

Let $P(x)$, defined in a sector with vertex at the origin, be analytic within the sector, in the neighborhood of the origin, and continuous on the sides of the sector, and at the origin. Let $P(x)$ have the asymptotic development of finite order

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \epsilon(x)x^n,$$

where $n \geqq 2$ and where $\epsilon(x)$ goes to zero with x. We say that

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* Since writing this note, I have ascertained by correspondence with Professor Birkhoff that he was familiar with and had used the result given here, but failed to publish it, being under the impression that it was contained in an article by W. B. Ford (*Bull. de la Soc. Math. de France*, 1911, p. 347).

$P'(x)$, the derivative of $P(x)$, which, of course, exists in the interior of the sector, exists also at the origin relative to any sector within the given one, and that in the inner sector $P'(x)$ has the development

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n-1)a_{n-1}x^{n-2} + \epsilon_1(x)x^{n-2}.$$

The proof is simple. Put $x = 1/y$. Then $P(1/y)$ is defined and is analytic, for sufficiently large values of y, in a sector S, which is very simply related to the sector of definition of $P(x)$, and has the asymptotic development

$$P(1/y) = a_0 + \frac{a_1}{y} + \frac{a_2}{y^2} + \cdots + \frac{a_n}{y^n} + \frac{\epsilon(1/y)}{y^n}.$$

We have

$$\frac{d}{dy}P(1/y) = -\frac{a_1}{y^2} - \frac{2a_2}{y^3} - \cdots - \frac{na_n}{y^{n+1}} - \frac{n\epsilon(1/y)}{y^{n+1}} + \frac{1}{y^n}\frac{d}{dy}\epsilon\left(\frac{1}{y}\right).$$

Consider now any sector S' interior to S, the sector of definition of $P(1/y)$. For sufficiently large values of y any point of S' is at least at a distance of unity from the sides of S. For any such point in S', we have

$$\frac{d}{dy}\epsilon(1/y) = \frac{1}{2\pi i}\int_c \frac{\epsilon(1/\zeta)d\zeta}{(\zeta - y)^2},$$

the integration being made around a circle of radius unity, so that

$$\left|\frac{d}{dy}\epsilon(1/y)\right| < M,$$

where M is the maximum modulus of $\epsilon(1/y)$ along the circle. Since $\epsilon(1/y)$ goes to zero as y becomes infinite, it is clear that we may collect the last three terms in the expression above for the derivative of $P(1/y)$, and write

$$\frac{d}{dy}P(1/y) = -\frac{a_1}{y^2} - \frac{2a_2}{y^3} - \cdots - \frac{(n-1)a_{n-1}}{y^n} - \frac{\epsilon_1(1/y)}{y^n},$$

where $\epsilon_1(1/y)$ goes to zero with $1/y$.

Reverting now to $P(x)$,

$$P'(x) = \frac{d}{dy}P(1/y)\frac{dy}{dx} = -y^2\frac{d}{dy}P(1/y)$$

$$= a_1 + \frac{2a_2}{y} + \cdots + \frac{(n-1)a_{n-1}}{y^{n-2}} + \frac{\epsilon_1(1/y)}{y^{n-2}},$$

and finally

$$P'(x) = a_1 + 2a_2x + \cdots + (n-1)a_{n-1}x^{n-2} + \epsilon_1(x)x^{n-2},$$

which is the asymptotic development sought. Also it results easily from the fact that $P'(x)$ approaches a_1 as x goes to zero that the derivative at the origin exists and is equal to a_1.

It is interesting to notice that a single differentiation has lost us two terms of the development of $P(x)$.* However, if $P(x)$ has an infinite asymptotic development, it is clear that $P'(x)$ will also have an infinite development, and in fact that the development of $P(x)$ can be differentiated formally any number of times.

The above proof applies, of course, to other domains than sectors; for instance, we might use the horn angle obtained by an inversion relative to the origin, and a reflection across the real axis of the infinite strip between any two parallel lines.

Lastly, it is important to notice that we have also established above the differentiability in the complex domain of asymptotic developments in descending powers of the variable.

COLUMBIA UNIVERSITY.

DARBOUX'S CONTRIBUTION TO GEOMETRY.

BY PROFESSOR L. P. EISENHART.

(Read before the joint session of the American Mathematical Society and the Mathematical Association of America at Cleveland, Ohio, September 6, 1917.)

GASTON DARBOUX was born in 1842 at Nîmes, a place of interest to mathematicians because here from 1819 to 1831 Gergonne edited his *Annales*, and incidentally exerted a great influence on the development of geometry. At the age of eighteen Darboux went to Paris, in whose intellectual life he had a prominent part for fifty-seven years. As a student, first at the Ecole Polytechnique and then at the Ecole Normale, his unusual mathematical ability made him conspicuous. His rare powers of expression soon won for him a reputation as a lecturer and expositor, with the consequence that he received desirable teaching posts from the beginning. In 1880 he succeeded Chasles in the chair of Géométrie supérieure at the Sorbonne. Four years later he became a Membre de l'Institut, and in 1889 assumed the duties of Doyen de la Faculté des Sciences. At the death of Bertrand in 1900 Darboux was elected Secrétaire Perpétuel de l'Académie des Sciences, which post he administered with distinction until his death. We cannot go further into detail concerning the many honors and responsibilities which came to him in his very full life. For we are concerned with the work he did in the field of geometry, to a review of which we now turn.

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* The loss of the term $na_n x^{n-1}$ is only apparent. This can be shown by taking the radius of the circle of integration above equal to $k\,|\,y\,|$, where k is some number independent of y. I prefer the proof above because of its applicability to more general domains than sectors.

While a student at the Ecole Normale in 1864, Darboux published his first papers, two notes in the *Nouvelles Annales*.* It is important that we consider these papers in some detail, as they contain germs of his subsequent work. The first dealt with the plane sections of the tore. It was shown that the curve is of the fourth order with the circular points at infinity for double points; that there are sixteen foci, in the sense of Plücker, of which four are real and lie on a circle; that there is a homogeneous linear relation between the distances of any point of the curve from three of these foci; and that an inversion with respect to any of these foci as pole transforms the curve into an oval of Descartes.

In his second note Darboux considered the curves of intersection of a sphere and a quadric. Through such a curve pass four quadric cones; the tangent planes of the cone meet the sphere in circles doubly tangent to the curve; the four of these planes tangent also to the sphere determine four null circles doubly tangent to the curve and lying on a circle; the distances of any three of them from any point of the curve are in homogeneous linear relation. The curve has, therefore, sixteen foci, lying in sets of four on four circles, such that any two circles meet orthogonally. To these curves and their transforms by inversion Darboux has given the name cyclics; they are spherical or plane and are of the fourth order. In particular, the plane sections of the tore and the ovals of Descartes are cyclics, as are also the cissoid, lemniscate, circular cubic, and other well-known curves.

In 1872 Darboux published his first treatise, entitled Sur une

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* Ser. 2, vol. 3 (1864), pp. 156, 199.

Classe remarquable de Courbes et de Surfaces algébriques et sur la Théorie des Imaginaires. It consists of five parts, one of which is devoted to a discussion of cyclics.

When Darboux was at the Ecole Normale, he became familiar with the writings of Lamé, Dupin, and Bonnet on triply orthogonal systems of surfaces, the field with which his name will ever be associated. At that time the best example of an orthogonal system of surfaces was afforded by the confocal quadrics. Years before Kummer had sought families of plane curves defined by an equation $f(x, y, a) = 0$, where a is the parameter of the family, such that through each point of the plane pass two curves of the family and that they meet orthogonally. He found that such families have the property that all the curves are confocal. Darboux, seeking to generalize this result, sought the triply orthogonal systems of surfaces, each of which can be defined by a single equation involving a parameter λ. In particular he sought the generalization of the orthogonal family of ovals of Descartes having three common foci. As a result he found the system defined by the single equation

$$
\begin{aligned}
(1)\qquad (x^2+y^2+z^2)^2+\frac{\alpha\lambda-4h}{\alpha-\lambda}x^2+\frac{\beta\lambda-4h}{\beta-\lambda}y^2 \\
+\frac{\gamma\lambda-4h}{\gamma-\lambda}z^2-h=0,
\end{aligned}
$$

where α, β, γ and h are constants; and λ is the parameter of the system. Each surface of the system so defined, subsequently called cyclides by Darboux, has the following properties: it is of the fourth order and has the imaginary circle at infinity for a double curve; it is cut by any sphere in a cyclic; in five different ways it is the envelope of a two-parameter family of spheres with their centers on a fixed quadric and orthogonal to a fixed sphere, each of the doubly tangent spheres meeting the surface in two circles. These results were presented to the French Academy of Sciences on August 1, 1864.* On the same day Moutard announced to the Academy the discovery of the same system. He had just been studying surfaces transformable into themselves by inversion, and had found that the surfaces of the fourth order with the circle at infinity for double curve are transformable in five different ways. While seeking the lines of curvature of these surfaces he found the triple system of cyclides.

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* *Comptes Rendus*, vol. 59 (1864), p. 240. The details of this paper were published in *Annales de l'Ecole Normale*, vol. 2 (1865), pp. 55–69.

After finding this orthogonal system, Darboux determined the linear element of space in terms of parameters referring to such a system of surfaces, and found, as in the case of confocal quadrics, that the curves of intersection form an isothermic system of curves on each surface. The celebrated theorem of Dupin, that in a triply orthogonal system any two surfaces of different families cut in a line of curvature for both, was made complete by the following addition due to Darboux:

When two families of orthogonal surfaces cut along the lines of curvature of these surfaces, there is a third family orthogonal to the first two families.

With the aid of this theorem Darboux was enabled to determine the condition to be satisfied in order that the family of surfaces defined by an equation of the form $\varphi(x, y, z) = \alpha$ can be part of a triple system; he found that φ must be any solution of a partial differential equation of the third order in two independent variables, but he did not calculate the equation, because of its complicated form. These results were published by Darboux in 1866 in his classic memoir "Sur les surfaces orthogonales"* subsequently presented as his thesis for the doctorate. This memoir contained also the determination of the orthogonal systems for which the lines of curvature are plane, and the erroneous theorem that the triple system of cyclides (1), which includes the system of confocal quadrics as a special case, affords the only example of a triple system of isothermic surfaces, a mistake corrected by Darboux in a later paper.

In 1872 Cayley attacked the problem of orthogonal systems, and gave the differential equation of triple systems the form now associated with his name, and valuable because of its adaptability to the determination of particular orthogonal systems. Darboux was quick to see the value of Cayley's work. On the one hand he made a study of the analytical processes underlying Cayley's equation and extended them to systems in n variables. On the other hand he made use of this equation for the determination of orthogonal systems for which the surfaces in one family are quadrics; those for which the surfaces in one family have a plane of symmetry; and orthogonal systems containing a given surface and involving in their equation four arbitrary functions of a single variable. These results were published in the first and second parts of his second great memoir on orthogonal systems.* In the third part he generalized the equations of Lamé to space of n dimensions and studied certain special problems in this connection. The last fifty pages of this memoir are devoted to the problem which he thought he had solved in his thesis, namely the determination of orthogonal systems of isothermic surfaces. He found other systems than those of cyclides, but we cannot go further into detail. As it is, we have devoted a great deal of space to these two memoirs, which, however, seems justified in view of their importance in the development of the theory of orthogonal systems, and of their place among Darboux's memoirs. Twenty years later he published his Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes, which includes much of the foregoing theory.† In the second memoir there appears also a reference to triply conjugate systems of surfaces, studied subsequently by Guichard and Tzitzeica and discussed at length by Darboux in the second edition of the above treatise issued in 1910.‡ In this volume and in subsequent papers Darboux made further contributions to the theory of orthogonal systems.

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* *Annales de l'Ecole Normale*, vol. 3 (1866), pp. 97–141.

From the time when Gauss introduced the idea of curvilinear coordinates of a surface and discovered the absolute invariant of the linear element which measures the curvature of the surface, geometers have been interested in the problem, as yet unsolved, of finding all surfaces with a given linear element, that is, the problem of applicable surfaces. In 1872 Darboux§ showed that the rectangular point coordinates of a surface with a given linear element are solutions of a partial differential equation of the second order of the Ampère type; also that when one solution of this equation is known two others can be found by quadratures, and that these three solutions are the coordinates of a surface with the given linear element.

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* *Annales de l'Ecole Normale*, ser. 2, vol. 7 (1878), pp. 97–151, 227–261, 275–349.

† A full analysis of this treatise, by E. O. Lovett, was published in this BULLETIN, vol. 5 (1899), pp. 185–202.

‡ A review by Wilczynski of this edition is published in the BULLETIN, vol. 20 (1914), pp. 247–253.

§ Sur une Classe remarquable, etc., pp. 14, 181.

Moutard and Ribaucour, who were at work on the problem of applicable surfaces in the late sixties, showed that when two applicable surfaces are known one can find at once two other surfaces S and S_1 whose rectangular coordinates x, y, z and x_1, y_1, z_1 satisfy the condition

$$dxdx_1 + dydy_1 + dzdz_1 = 0. \tag{2}$$

In 1873 Darboux announced to the Société mathématique de France* that the problem of the infinitesimal deformation of a surface S is equivalent to the determination of the surfaces S_1 whose coördinates satisfy equation (2). In his well-known treatise Leçons sur la Théorie générale des Surfaces† he showed that when the equation (2) for a surface S has been integrated, the infinitesimal deformation of S, to terms of any order, is a problem of quadratures alone. Weingarten‡ reduced the solution of equation (2) to the integration of a linear partial differential equation of the second order, whose equation of characteristics defines the asymptotic lines of the given surface. When the latter are parametric, the equation is of the form

$$\partial^2\theta/\partial u\partial v = \mu(u, v)\theta. \tag{3}$$

A surface S_0 corresponding to a given surface in such a way that the tangent planes to the two surfaces at corresponding points are parallel and asymptotic lines on each surface correspond to a conjugate system on the other is said to be associate to it. This idea is due to Bianchi,§ who proved that the determination of the surfaces associate to a given surface is equivalent to the integration of equation (2). Having noted the reciprocal character of the relations between S and S_1 and between S and S_0, Darboux found nine other surfaces arising from each solution of equation (2), the whole group of twelve surfaces forming a closed system.

It would be a great oversight were we not to refer to the charming chapters in the fourth volume of the Leçons in which Darboux treats the rolling of one applicable surface upon another, and the geometrical configurations thus generated by points, lines, etc., invariably fixed with respect to the rolling surface. Ribaucour also had a part in the development of this field and recently Bianchi. As an example of the beauty of some of the theorems in this theory we cite the following: Let S and S_1 be two applicable surfaces; as S rolls over S_1, a null sphere with center invariably fixed with respect to S cuts the common tangent plane to the two surfaces in a circle; these circles form a cyclic system.

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* This paper was not published.

† Vol. 4, p. 5. Hereafter this treatise will be referred to as the Leçons.

‡ *Crelle*, vol. 100 (1886), pp. 296–310.

§ Lezioni di Geometria differenziale, Pisa, 1894, p. 279.

The idea of the spherical representation of a surface is due to Gauss. The lines of curvature of a surface are represented on the sphere by an orthogonal system of curves. Darboux early interested himself in the solution of the problem of finding the surfaces whose lines of curvature are represented on the sphere by a given orthogonal system, which he called the "problem of spherical representation." By making use of the simple expressions for the rectangular coordinates of a point of the sphere in terms of parameters referring to the imaginary generators, he showed that the problem is reducible to the solution of a partial differential equation of the form (3). He wrote a number of papers dealing with special forms of this equation, and with the geometrical significance of this work and of the researches of Moutard on the equation (3).

We recall that Moutard showed that a cyclide of the fourth order is, in five different ways, the envelope of a two parameter family of spheres orthogonal to a fixed sphere S, the centers of the spheres being on a quadric; that the five quadrics thus associated with the cyclide are confocal; and that any two of the five fixed spheres S cut one another orthogonally. When he announced his discovery of the orthogonal system of cyclides, Moutard remarked that for all the cyclides of the system these five spheres and five quadrics are the same. Making use of this result Darboux showed* that the equation of the system can be written in the form

$$\sum_{i=1}^{5} \frac{\left(\frac{S_i}{R_i}\right)^2}{\lambda - a_i} = 0, \tag{4}$$

where S_i is the power of the point with respect to the sphere S_i, R_i is its radius, a_i is a constant and λ the parameter of the system. This equation being necessarily of the third order the coefficient of λ^4 is identically zero. Hence we have the

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[*] Sur une Classe remarquable, etc., p. 134. The fourth and fifth parts of this treatise are devoted to an exposition of the geometry of cyclides.

identity

$$\sum_{i=1}^{5}\left(\frac{S_i}{R_i}\right)^2 = 0 \tag{5}$$

for any point in space. Darboux saw in these quantities S_i/R_i, subject to the above condition, a new type of coordinate which could be used to good purpose in certain problems. Thus were discovered *pentaspherical coordinates*. An important theorem involving them is the following:

If five particular solutions $x_1, \cdots, x_5$ of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v} + C\theta$$

satisfy the relation $\Sigma x_i^2 = 0$, the quantities x_i are the pentaspherical coordinates of a surface upon which the parametric curves are the lines of curvature. When, in particular, the above equation is reducible to the form (3), the surface is isothermic. Important consequences of this result have been obtained by Darboux and Guichard.

From the equations of Gauss it follows that the non-homogeneous coordinates x, y, z of a surface, referred to a conjugate system of curves $u =$ const., $v =$ const., satisfy an equation of the Laplace form

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{6}$$

called the point equation of the system. However, Darboux called attention to the fact that each function $\varphi(x, y, z)$, in general, determines a conjugate system, found by the solution of an ordinary differential equation of the first order and second degree, in the sense that the corresponding equation (6) admits the solution φ as well as x, y, and z. In particular, the lines of curvature of any surface are characterized by the property that they form the conjugate system for which $\varphi \equiv x^2 + y^2 + z^2$. Guichard has extended this conception and considered the remarkable class of conjugate systems n, O, defined by the property that the point equation of such a system admits $n - 1$ solutions $t_1, \cdots, t_{n-1}$ and that $x^2 + y^2 + z^2 - t_1^2 - \cdots - t_{n-1}^2$ also is a solution.

Darboux has been such a conspicuous worker in differential geometry that one may make the mistake of thinking that his geometrical work has been confined to this field. To offset this impression we recall that the greater part of his treatment of cyclics and cyclides is not differential. We mention also his studies of Poncelet polygons, and their generalization to polygons inscribed and circumscribed to ellipsoids, and some of his researches on the wave surface. It is an interesting fact that his first treatise, previously referred to, and his last, Principes de Géométrie analytique, just published,* deal almost entirely with finite geometry.

In 1868 Beltrami published† his striking interpretation of the geometry of Lobachevsky, Bolyai and Gauss in terms of geodesics on a pseudospherical surface. Three years later Klein‡ pointed out the significance for non-euclidean geometry of the concept of an absolute and of the associated metric which Cayley had developed in his Sixth Memoir on Quantics, twelve years before. These new ideas made a decided impression on Darboux. He made use of the Cayley measurement in the theory of cyclides as presented in his first treatise, and in a note appended to this volume he studied geodesics and lines of curvature in Cayleyan geometry. These ideas are developed more fully in the last chapter of the third volume of his Leçons, and are applied there to the determination of surfaces upon which there is a conjugate system of curves all of whose tangents are tangent to a quadric also. One of the five parts of the Principes de Géométrie analytique is devoted to a very clear exposition of Cayleyan geometry, displacements in this geometry and its trigonometry. But for one reason or another Darboux confined his researches to euclidean space, and he had no part in the development of the theory of non-euclidean differential geometry.

Darboux was a strong advocate of the use of imaginary elements in the study of geometry. He believed that their use was as necessary in geometry as in analysis. He had been impressed by the success with which they had been employed in the solution of the problem of minimal surfaces. From the very beginning he made use in his papers of the isotropic line, the null sphere (the isotropic cone) and the general isotropic developable. In his first memoir on orthogonal systems of surfaces he showed that the envelope of the surfaces of such a system, when defined by a single equation, is an isotropic developable. We have given examples of the use of these elements in the theorem cited for rolling surfaces, and likewise in the solution of the problem of spherical representation. Another striking example is afforded by the following theorem, which gives a general method of obtaining surfaces with plane lines of curvature in one system: If a developable D rolls over a developable D_1 in such a way that all the points of a generator are in contact at any instant, an isotropic developable invariably fixed with respect to D will be cut by successive planes of contact along lines of curvature of the surface which is their locus.

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* This treatise is in fact an enlarged edition of the first. The writer will shortly publish a detailed review of it.

† *Giornale di Mathematiche*, vol. 6 (1868), pp. 284–312.

‡ "Ueber die sogennante Nicht-Euklidische Geometrie," *Math. Annalen*, vol. 4 (1871), pp. 573–625.

In a discussion of so prolific a writer as Darboux one can merely mention the results of some of his investigations and cannot give an idea of the details. However, it would be a mistake in his case not to refer to the character of the methods used. Darboux gives to Combescure the credit of being the first to apply the considerations of kinematics to the study of the theory of surfaces with the consequent use of moving coordinate axes. But to Darboux we are indebted for a realization of the power of this method, and for its systematic development and exposition. This exposition is to be found in the first two volumes of his Leçons, with applications later to the discussion of particular types of surfaces and of orthogonal systems. The method has been used extensively by his students and followers, and it is sure to be used by any worker in the field of differential geometry who is familiar with its processes.

Darboux's ability was based on a rare combination of geometrical fancy and analytical power. He did not sympathize with those who use only geometrical reasoning in attacking geometrical problems, nor with those who feel that there is a certain virtue in adhering strictly to analytical processes. His geometrical proofs of the theorems dealing with rolling surfaces referred to above are as pure as they are simple and beautiful. No less brilliant are his reductions of various geometrical problems to a common analytical basis, and their solution and development from a common point of view. At times in the solution of a problem we find a combination of the two methods. This is seen in various parts of his treatises, and in particular in the memoir in which he solved the problem of finding the two-parameter families of spheres for which the correspondence between the two sheets of the envelope is conformal.* By geometrical considerations it is shown that the lines of curvature on the two sheets are in correspondence, and then by analysis that they are isothermic surfaces. Here is established the transformations of isothermic surfaces, which Bianchi has called the transformations D_m, and the associated deformations of quadrics. However, we believe that Darboux's interest lay rather in analytical geometry. This was natural in one who possessed such remarkable powers as an analyst. We have not the time to discuss, nor indeed to enumerate, the contributions to analysis which Darboux made while in pursuit of the solution of geometrical problems. Surely we find realized in his works his belief that "the alliance between geometry and analysis is useful and productive; perhaps this alliance is a condition for the success of both."

In the foregoing sections we have tried to give an idea of the scope of Darboux's contribution to geometry. His writings possess not only content but singular finish and refinement of style. In the presentation of results the form of exposition was carefully studied. Darboux's varied powers combined with his personality in making him a great teacher, so that he always had about him a group of able students. In common with Monge he was not content with discoveries, but felt that it was equally important to make disciples. Like this distinguished predecessor he developed a large group of geometers, including Guichard, Koenigs, Cosserat, Demoulin, Tzitzeica, and Demartres. Their brilliant researches are the best tribute to his teaching. His spirit will continue to live in these men, and in the many who will turn to his works for direction and inspiration. Hence of Darboux also the prophecy may be true which Lagrange made of Monge, when he said "With his geometry this devil of a man will make himself immortal."

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* "Sur les surfaces isothermiques," *Annales de l'Ecole Normale*, ser. 3, vol. 16 (1899), pp. 491–508.

CREMONA'S WORKS.

Opere matematiche di Luigi Cremona. Pubblicate sotto gli auspici della R. Accademia dei Lincei. Volumes 1 and 2. Milano, Hoepli, 1914 and 1915. Quarto, viii + 492, 451 pp.

CREMONA'S career as geometer and teacher covered very nearly the second half of the nineteenth century. Born at Pavia in 1830, he was only eighteen when his ardent patriotism drew him into the war of independence. Returning after a year and a half, at the close of the war, he studied at the University of Pavia under Bordoni and Casorati and in 1853 took the laureate in civil engineering and architecture. After seven years of teaching in lower schools, he was called in 1860 to the University of Bologna as first professor of projective geometry and mechanics. After six years of intense activity there, he returned to Milan as a colleague of Brioschi at the Polytechnic and Normal School, training teachers in graphical statics for the technical institutes of the new Italy. From 1873 until the end of his life, 1903, he lectured in the University of Rome, on geometry, graphical statics, and "higher mathematics," meanwhile giving time and care without stint to the school of engineering, of which he was the founder and director. Usually also he gave courses in the normal department, to which he attached no less importance than to the more purely theoretical studies.

It may be doubted whether any great teacher has been actuated primarily by considerations of economic utility. While Cremona was intensely patriotic, it is evident from his writings that it was the innate love of his chosen science that moved him to teaching and to the preparation of the books through which his name is most widely known. In this collection the editors have not included his Elements of Projective Geometry; but we have his Introduction to a geometrical Theory of Plane Curves (1862), which was later translated (1865) into German by Curtze, and attained wide circulation and use; also his Fundamentals of a geometric Theory of Surfaces. This latter, combined with the memoir on Cubic Surfaces, was also most extensively known in Curtze's German version (1869). Here too are included, of course, the two epoch-making essays on transformations of plane curves (1863, 1865). It will be noticed that all these appeared during the Bologna period of six years. By way of comparison we note that Salmon's Higher Plane Curves first appeared in 1852, and that most of Steiner's work on algebraic curves was published between 1848 and 1854.

These two volumes* contain 78 titles, covering the author's publications down to 1868, from Pavia, Milan, and Bologna. Of these 78, 45 are dated from Bologna. Nine in the first, seven in the second volume, are answers to questions proposed in the *Nouvelles Annales*. Eight are book reviews, and neither dull nor devoid of significance; four are historical articles or addresses intended to stimulate geometrical study and research. One should read first of all the short peroration (volume 1, pages 252–3) of the inaugural address to his course on higher geometry at Bologna. Here occurs the often quoted climactic injunction: "Credite all' avvenire! questa è la religione del nostro secolo." But of more value to the young mathematician is the admonition: "Senza un' incrollabile costanza nella fatica non si giunge a possedere una scienza. Se questo nobile proposito è in voi, io vi dico che la scienza vi apparirà bella e ammiranda, e voi l'amerete così fortemente che allora in poi gli studi intensi vi riusciranno una dolce necessità della vita." There are not many such valuable sources of information and inspiration accessible to ambitious students.

Chasles was the type on whom at first Cremona modeled his own work. The geometrical purism of von Staudt was a later influence. Hence it is natural to find metric foundations for geometric definitions instead of, or interchangeably with, projective. The apparatus of algebraic geometry is built upon polars, and these upon distances. The geometric method is principally a use of terms or descriptive relations instead of equations. And the beginnings of enumerative geometry are here: we find questions of intersections or determining conditions for curves or surfaces treated by what has since become the principle of the Erhaltung der Anzahl, the axiom that the number persists even when the loci involved become specialized or degenerate. What Cremona is trying to attain and to diffuse is, not a critical knowledge of foundations, but an extensive knowledge of objects and theorems in projective geometry. This will doubtless be the path followed by students for a long time to come; for it is easier to analyze and to discriminate than to unite discrete theories.

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* The third and concluding volume has been published more recently (1917).

The Introduction contains an account quite complete, at its date, of plane cubic curves. It was to be desired, as Curtze said, that the theory of surfaces should cover those of the third order, and for this there was ample material in Cremona's Steiner prize essay. The depiction of a general cubic surface upon a plane, its rationality, was discovered independently by Cremona and Clebsch, and enabled the former to work out fully the geometry of algebraic curves upon the surface. This theme is more elaborately studied by R. Sturm in his well-known paper in the *Mathematische Annalen*, volume 21 (1883). A footnote there might suggest the notion that Cremona had committed a serious error in mentioning only one out of nine possible cases where the intersection of a cubic with a sextic surface breaks up into two twisted nonic curves. A reference to Cremona's own statement shows, however, that he was not attempting an exhaustive list of cases. This particular statement, like many in the same chapter, concludes with "etc., etc." That derogatory criticism was not in Sturm's intention is apparent also from the opening section of his "Nachruf" for his friend (*Archiv für Mathematik und Physik*, series 3, volume 8, 1904, pages 11–29).

In this Nachruf and in two others, Noether's in *Mathematische Annalen*, volume 59 (1904), pages 1–19, and Bertini's in *Proceedings of the London Mathematical Society*, series 2, volume 1 (1903–4), pages v–xviii, is found a clear and careful analysis of most of Cremona's important scientific papers. In particular, Sturm very neatly and fully analyzes the series of papers on twisted cubic curves, ten in number, which appeared from 1858 to 1864. The first two were devoted mainly to proving theorems, some twenty-five in number, which Chasles had published without proof in a note to his Aperçu historique, but both these and the later ones contain much that was original. The three that Chasles gives upon loci arising in the infinitesimal motion of a rigid body Cremona seems not to consider important enough to reproduce, as indeed they are but restatements of abstract theorems. The erroneous theorem of Chasles he does not criticize nor correct, namely that the locus of vertices of quadric cones passing through six arbitrarily fixed points in space is a twisted curve of the third order, whereas it is obviously two-dimensional and actually a Weddle's quartic surface.

One of these papers on twisted cubics had the experience, not unusual before the advent of the *Jahrbuch über die Fortschritte der Mathematik*, of being forgotten, and later duplicated. Böklen and F. Meyer in 1884 studied the question as new: When has a cubical parabola a directrix? If every osculating plane is perpendicular to two others that meet at a right angle, the locus of their common point is termed a directrix. When there is a directrix, what is its nature? The cubical parabola is a twisted cubic which osculates the plane at infinity, so that its other osculating planes trace out at infinity a conic. Orthogonality is conjugateness with respect to the absolute, an imaginary circle at infinity; so that the question concerns polar triangles of one conic circumscribed to another. These matters, and others related to them, had been thoroughly discussed in Cremona's final paper on twisted cubics in 1864, No. 50 of the present collection, particularly in § V. This fact probably escaped the notice even of Schroeter, whose writings were contemporaneous; for he does not mention it in his highly interesting paper: "Metrische Eigenschaften der cubischen Parabel," in *Mathematische Annalen*, volume 25 (1885). Such instances as this, of truths discovered yet insecurely fixed in the body of science, help one to realize the value of the Encyklopädie der mathematischen Wissenschaften.

Cremona himself was conscientious and indefatigable in searching out the work of his predecessors upon matters that he himself was investigating. Witness for example the voluminous list cited in his review of the translation of Amiot's geometry. This virtue is not too common, and we may be permitted to quote from Poncelet, who published with pardonable pride in volume 1 of his Traité (page 420) a letter from Dupin containing this tribute to that kind of honest dealing. "Je vous loue beaucoup d'avoir aussi rappelé honorablement les travaux de tous vos prédécesseurs; croyez-moi, cela n'ôte rien à votre mérite, et donne une haute idée de votre caractère. Vous prouvez par là que vous n'avez rien de commun avec cette école égoïste qui voudrait faire un monopole de la célébrité mathématique. Qu' ils maigrissent à leur gré de l'embonpoint d'autrui, et faites-les maigrir." Apropos, one reads with admiration the footnote

(volume 2, page 18) wherein Cremona, after two pages of historical summary, excuses his previous ignorance of, and failure to cite, the works of Möbius and Seydewitz to which an essay of Schroeter had now directed his attention. He concludes with the phrase: "A présent je restitue unicuique suum." To be just is not easy, but he believed it to be a duty.

Of all these valuable works which constitute the visible monument of a valiant geometrician, those which will longest secure his fame are those numbered 40 and 62 in volume 2, upon geometric transformations of plane figures. Beyond the linear or projective transformations of the plane there were known the quadric inversions of Magnus, changing lines into conics through three fundamental points and those exceptional points into singular lines, to be discarded. Cremona described at once the highest generalization of these transformations, one-to-one for all points of the plane except a finite set of fundamental points. He found that it must be mediated by a net of rational curves; any two intersecting in one variable point, and in fixed points, ordinary or multiple, which are the fundamental points and which are themselves transformed into singular rational curves of the same orders as the indices of multiplicity of the points. When the fundamental points are enumerated by classes according to their several indices, the set of class numbers for the inverse transformation is found to be the same as for the direct, but usually related to different indices. Tables of such rational nets of low orders were made out by Cremona and Cayley, and a wide new vista seemed opening (such indeed it was and is) when simultaneously three investigators announced that the most general Cremona transformation is equivalent to a succession of quadric transformations of Magnus's type. This seemed a climax, and a set-back to certain expectations. But the fact remained, that elements do not constitute a theory, and that the generators of a group are no more important than its invariants. These invariants, for the most part, are yet to be determined, both synthetically and algebraically.

The expectation of geometrical inventors was turned next upon space of three dimensions; and there Cremona showed how a great variety of particular transformations can be constructed, but anything like a general theory is still in the future. When found, if within this century, such one-to-one transformations of three-dimensional space are certain to be hailed as Cremona transformations.

It is honorable alike to the Royal Academy of the Lincei and to the colleagues of Cremona and younger mathematicians, that they unite to preserve in worthy form the works of a justly celebrated scientist and leader. Eighteen already have shared the not inconsiderable labor of thorough editing, and their corrections and explanatory notes, appended to each volume, form a valuable aid to the reader. The highest tribute that can be paid to the memory of a scientist is the labor that makes his work more useful to the next generation.

HENRY S. WHITE.

BLICHFELDT'S COLLINEATION GROUPS.

Finite Collineation Groups. By H. F. BLICHFELDT. Chicago, University of Chicago Press, 1917. 12mo. 12 + 194 pp.

THIS little volume forms a notable contribution to the series of mathematical texts by American authors that have appeared in recent years. Coming from the pen of an author who has an unusual mastery of his subject, it is moreover almost unique in its field, the promised text by Wiman for the Teubner series (as far as the reviewer is aware) not having appeared. Certain parts of the subject, particularly the theorems depending on the invariance of a Hermitian form and the theory of group characteristics, may be found in the second edition of Burnside's Theory of Groups, which appeared in 1911. A considerable part also of the material in the present treatise may be found in Part II of Finite Groups, by Miller, Blichfeldt, and Dickson, which was written by the same author.

On the other hand there is much in the present volume that cannot be found elsewhere except in scattered journal articles, and some of the results at the close of Chapter IV seem to be entirely new. The author's own share in the development of the subject is a very notable one, the theorems in Chapter IV concerning the linear groups in n variables being almost entirely his own. In addition the complete determination of the groups in three and four variables was first made by him, the earlier work along this line being reproduced in Chapters V and VII in a somewhat revised form. There are in addition chapters on the "Elementary Properties of Linear Groups," "Groups of Operators and Substitution Groups," "The Linear Groups in Two Variables," "The Theory of Group Characteristics," and "The History and Applications of Linear Groups."

As the author remarks in the closing chapter, the theory of linear groups may be said to have originated with Klein, who was mainly interested in their application to the solution of algebraic equations. At about the same time, however, they were employed by several writers, notably Schwarz, Fuchs, and Jordan, in the study of linear differential equations having algebraic integrals. The theory received in this way such an impetus that it has now grown to considerable proportions.

The problem of the determination of the finite collineation groups has been attacked in two ways. One may ask either "What groups may be represented in a given number of variables?" or "In how many ways may a given abstract group be represented as a linear group?" Concerning the first of these questions there is the theorem of Jordan that the order of a finite linear group in n variables is of the form λf, where f is the order of an abelian self-conjugate subgroup, and where λ is inferior to a fixed number that depends only upon n. The theorems of Chapter IV enable the author to obtain a definite limit for λ; in fact, he was the first to obtain such a limit, although other (much higher) limits have been obtained by Bieberbach and Frobenius. The exact values that λ may have are known only for $n = 2, 3, 4$.

The theory of characteristics, which is due mainly to Frobenius, throws considerable light on the second question. One of the theorems obtained by this means is to the effect that if a given abstract group of order g be represented as a "regular" group of permutations on g variables, and then by means of a change of variables the group be represented by means of a series of "transitive" component groups, every possible representation of the given abstract group as a transitive linear group will occur in this series and the number of times it occurs is equal to the number of variables in which it is represented. Moreover, the number of such representations that are "non-equivalent" is equal to the total number of sets of conjugate operators of the given abstract group.

The author includes these theorems in his chapter on the theory of group characteristics. It seems unfortunate, however, that he should have used the terms "equivalent or non-equivalent groups" instead of "equivalent or non-equivalent representations of a group." This point will be discussed more in detail later.

There seems no doubt that the author was desirous of developing the subject of linear groups proper as far as space would permit, and for this reason has omitted any extended discussion of the geometrical properties of the individual groups, such as their invariants. The methods used in the determination of the groups in three and four variables are as a rule analytical, although geometrical arguments have been employed in a few places.

In spite of the fact that the treatment was evidently not intended to be exhaustive, the reviewer was somewhat disappointed not to find anything concerning the groups that are known to exist in more than four variables. Perhaps the most interesting groups of this sort are the three systems in p^m, $(p^m - 1)/2$, and $(p^m + 1)/2$ variables that are isomorphic with the "abelian linear group" on $2m$ indices, the coefficients of which are residues, modulo p. For $m = 1, 2$ these groups have been considered by Klein and several others in their relation to elliptic and hyperelliptic functions. The ternary groups of order 216, 168, 60 and the quaternary groups of order 11520, 25920, 168 each belong to one of these systems. There are in addition groups in six, seven, and eight variables, reference to which might have been included among the historical notes in the closing chapter.

In taking up the consideration of the various chapters somewhat more in detail, special mention should be made of the fundamental theorem, due to several authors, that every finite linear group in n variables has an invariant definite Hermitian form. This theorem, which is included among those given in the first chapter, is used to establish the result, due originally to Maschke and later extended by Loewy, that every "reducible" group is "intransitive." Thus, for example, every collineation group in three-dimensional space that has an invariant plane must also leave fixed a point not in that plane. This result is used in turn to show that every linear group in which the transformations are commutative may be written in canonical form. It might perhaps have been well to direct the reader's attention to the fact that this is not necessarily the case for collineations.

The second chapter contains an introduction to the theories of abstract groups and permutation-groups, introduced mainly for the benefit of readers who may have had no previous acquaintance with the subject. Under the former heading we find for example Sylow's theorem and some discussion of abelian groups and groups having for order a power of a prime. With regard to permutation groups there is included something concerning the representation of a given abstract group as a "regular" group and some theorems on the alternating and symmetric groups.

For the determination of the linear groups in two variables the author has chosen on account of its historical and geometrical interest the process employed by Klein, whereby the possible groups are shown to be isomorphic with the groups of rotations of the regular solids. An outline is also given of Jordan's process for finding these groups by means of a diophantine equation. Another somewhat similar equation which would answer the same purpose may be obtained from Theorem 4 of the chapter on group characteristics. It follows from this that the order g of any transitive collineation group in two variables must satisfy the equation

$$g = 4 + g\sum_{i} f_i(n_i - 2)/n_i,$$

where the summation is to be taken over the orders $n_1, n_2, \cdots$ of the cyclic subgroups in the different conjugate sets, and each $f_i = 1, 2$.

The fourth chapter is the one that is likely to attract the widest attention. One of the most interesting theorems that it contains is to the effect that the order of a primitive group in n variables cannot be divisible by a prime greater than $(n - 1)(2n + 1)$, a result first published by the author in the *Transactions* in 1903. Whether for large values of n there exist groups whose orders are divisible by primes nearly as large as this seems rather doubtful; in fact, as far as the reviewer is aware, no primitive group is known to exist whose order is divisible by a prime greater than $2n + 1$. It is a real achievement, however, to have been able to obtain any limit at all.

In the analysis employed in the proof a certain equation involving only roots of unity is shown to exist connecting the characteristics of certain transformations. By use of Kronecker's theorem concerning the irreducibility of the general cyclotomic equation the author ingeniously transforms this equation into a congruence, from which it follows that the product of any two transformations of order p, a prime exceeding the limit given above, must be either of order p or identity. For special values of n he is able to reduce this limit. For $n = 3, 4$ it is found that no primes exceeding 7, 13 respectively can divide the order of a primitive group.

The author's statement of Kronecker's theorem (§133), or rather the consequence that he deduces from it, is not quite accurate. For example, the rational equation of lowest degree that is satisfied by θ, a primitive 15th root of unity, is

$$\theta^8 - \theta^7 + \theta^5 - \theta^4 + \theta^3 - \theta + 1 = 0,$$

but the seven roots of unity appearing in this sum cannot be separated into sets of the sort that he describes. This does not form a serious defect in his argument, however, since it does follow from Kronecker's theorem that every sum of roots of unity that is equal to zero may be thrown into the form (4), page 85, by the addition and subtraction of roots that cancel each other by pairs. The second paragraph under 7°, §133, is subject to a similar modification.

There are also theorems that limit the conditions under which a primitive simple group can contain transformations having for order a power of a prime and commutative transformations of different orders. In Theorem 7, page 93, condition (B) is unnecessary, since there do not exist more than m different roots of unity whose mth powers are all equal to the same root.

In the latter part of the chapter an interesting type of analysis first employed by Bieberbach and Frobenius is used by the author to show that no primitive group can contain a transformation whose multipliers when represented graphically on the unit circle all lie on an arc that does not extend more than 60° on either side of some one of them. This theorem is found useful in the chapters on the groups in three and four variables. It is also used in the section that immediately follows to establish an upper limit for the order of any abelian subgroup that can be contained in a primitive group in n variables.

Finally by use of the various theorems of the chapter a limit is obtained for the order of a primitive group in n variables, a result that has been referred to above. This limit is lower (as far as the reviewer is aware) than any that have been published previously. The two theorems moreover that were referred to in the preceding paragraph seem also to be new, although the first of these bears a close relation to one obtained by Frobenius.

In the succeeding chapter these results are applied to the determination of the primitive groups in three variables. The first accurate solution of this problem was obtained by the author of this text, the previous attempts by Jordan and Valentiner being only partially successful. These groups are found to be the "Hessian" group of order 216, so called by Jordan on account of its relation to the inflexional points of a pencil of cubic curves, two of its subgroups of orders 72 and 36, and three simple groups of order 60, 168, 360. The last two were first found in the order named by Klein and Valentiner.

Although a number of expositions of the theory of group characteristics have been given, notably by Frobenius, Burnside, Schur, and Dickson, the author does not follow closely any of them. In making essential use of the invariance of a Hermitian form his treatment probably resembles Burnside's more than any of the others.

Some of the theorems here give additional information concerning the possible primitive groups in n variables. By means of one of them it follows that if any group contains transformations of orders p and q, two distinct primes each greater than $n + 1$, it must also contain transformations of order pq. As far as the reviewer is aware, no primitive group of this sort is known to exist. Another useful relation is that for a transitive group the sum of the products of the characteristics and their conjugate imaginaries (for each of the transformations) is equal to the order of the group. An application of this theorem to the determination of the binary groups has been mentioned above.

Some of the theorems concerning the representation of a given abstract group as a linear group have already been referred to. To illustrate the sense in which the author uses the terms "equivalent or non-equivalent groups," we consider the linear group of order 27 generated by $x' = y$, $y' = z$, $z' = x$, and $x' = x$, $y' = \omega y$, $z' = \omega^2 z$, where $\omega^2 + \omega + 1 = 0$. We may put the transformations of this group into (1, 1) correspondence with those obtained by replacing ω by ω^2, which latter form therefore a group simply isomorphic with the first. According to the author's terminology (Theorem 5, page 128) these two groups are "non-equivalent," doubtless for the reason that it is impossible by a change of variables to make the *corresponding* transformations simultaneously identical. On the other hand they contain exactly the same transformations in the form in which they are written, though arranged in a different order.

This seems to the reviewer an unusual use of these terms. As the author apparently does not mean the definition on page 64 to be taken in this sense, it seems not unlikely that the reader may place a wrong interpretation on some of the theorems in this chapter. That this is the meaning that must be attached to the word "equivalent" may be seen from a careful examination of the proof of Theorem 5.

An interesting application of the theory of characteristics is made at the close of the chapter to prove the well known theorem that every group of order $p^a q^b$ is composite.

In Chapter VII the author applies the theorems of Chapters IV and VI to the determination of the primitive groups in four variables. The exposition of the subject in this treatise has been somewhat modified from the original one, which appeared in the *Mathematische Annalen* in 1905, and which was an achievement of no mean order. If any primitive group is not itself simple (as a collineation group), then somewhere in its chain of factor groups there will occur either a primitive simple group or else a group that is not primitive, thus affording a basis for a classification.

It follows at once from geometrical considerations and by use of the knowledge concerning the groups in two and three variables that no primitive group in four variables can contain transformations with two pairs of equal multipliers of higher order than 5 or transformations with three equal multipliers (homologies) of higher order than 3. In a separate discussion it is shown that transformations of the former type having the order 5 may also be ruled out. The simple group of order 25920 is found to be the only primitive group that contains homologies of order 3.

The discussion at the bottom of page 145 may not be entirely clear to a reader. The author remarks that the two binary groups of order $60 \cdot 2$ must be "equivalent," since the two generating transformations of order 5 have the same multipliers in the two cases, the term being used doubtless in the same sense as in the preceding chapter. It does not seem evident without further consideration that the quaternary group might not be (2, 1) isomorphic with either of the two binary groups, in which case Theorem 5 of Chapter VI would not give the desired result. This is apparently the theorem that the author is using here, although it is not explicitly referred to.

The proof of Theorem 3 would have been much simplified if after establishing the existence of a subgroup of order 648 the author had used the fact that any two homologies of order 3 that are not commutative must generate a group of order 24 in which all four homologies are conjugate and which contains an invariant reflection. Thus a homology must be conjugate with any that are not commutative with it and hence in a primitive group with all others. Also any homology not in the subgroup of order 648 must be commutative with one of the nine reflections that this subgroup contains, from which it follows that there cannot be more than 40 homologies altogether. By a proper choice of coordinates one of the nine reflections commutative with "*D*" may be taken as "*F*."

In view of Maschke's theorem (page 23) and the fact that there is no larger collineation group in three variables containing a G_{216}, the argument on page 151 beginning, "We now write down, etc.," appears superfluous.

Having eliminated from consideration certain types of transformations the author is able to show that the possible Sylow subgroups that may occur in a primitive simple group are of a comparatively limited number of types. A number of the theorems of Chapters IV and VI are found useful. One device that is employed consists of the introduction of line coordinates and the application of some of these theorems to the resulting group in six variables. In the reviewer's opinion it would have been desirable to have given some explanation of the method of excluding transformations of order 9. The final conclusion is that the only primitive simple groups that exist are of order 60 (two types), 168, 360, 2520, 25920.

The only groups in this list that can be contained self-conjugately in larger groups are those that are isomorphic with the alternating groups on five or six letters. Each of these is contained by a group of twice its order isomorphic with the corresponding symmetric group. These in turn are not contained self-conjugately by any larger groups.

The primitive groups containing invariant intransitive subgroups are all found to leave invariant a quadric surface. Some of these can be enlarged by the addition of transformations that interchange the two sets of rulings. Finally it is shown that the only primitive groups that contain imprimitive invariant subgroups are the one of order 11520 and certain of its subgroups.

The final chapter on the history and applications of linear groups has been referred to above. The author gives an interesting sketch of the methods used in their application to the solution of algebraic equations and describes also their relation to linear differential equations having algebraic integrals.

The book appears in a neat, attractive form and the proofreading has been carefully done, only a few errata being found. Those that were noticed, together with some minor comments, are as follows:

P. 19 (next to the last line). Replace $\bar{x}_{n-1}$ by $\bar{x}_n$.

P. 20 (8th line). Replace one y_{n-1} by $\bar{y}_{n-1}$.

P. 20 (19th line). Replace f'_0 by f.

P. 65 (next to last line). The value of Y should be $Y = x_1\bar{x}_2 + x_2x_1$.

P. 72. The numbers given here are the numbers of *rotations* rather than the numbers of the *axes*. There are only 10 different axes of period 3 and 6 of period 5.

P. 79 (7th line). Replace Y_2'' by Y_1''.

P. 90 (3d line from bottom). Replace $p_a \gtreqless kp$ by $p^a \geqq kp$.

P. 100. It is perhaps worth while to remark that the sets of intransitivity "of highest index" referred to at the middle of the page may not coincide with the "ultimate" sets of intransitivity referred to in the preceding discussion.

P. 110. The author apparently intended that T' should be of period 4. For this we may take $\alpha = 1$, $\beta\gamma = -1$.

P. 112 (6th line). Replace $2^3 \cdot 3^2 \cdot \phi$ by $2^3 \cdot 3^2$.

P. 113 (last line). A factor 1/7 has been omitted from the first expression for h.

P. 126 (middle). The expression in the second parenthesis should be $Y_2 + a_2Y_3$.

P. 154. The orders of Q for the groups (g), (j), (k) should be respectively 10ϕ, 75ϕ, 150ϕ.

In view of the unusual conditions prevailing at the present time the book will undoubtedly not attract the attention it deserves. It ought to serve however to stimulate interest in the subject at least in this country. There are few, if any, theories that are possessed of greater elegance or that offer a more direct challenge to the mathematical investigator.

As a text for class-room use it will be found very suitable except perhaps for those whose interests center mainly in the geometrical aspects of the subject. Some of the analysis in Chapter IV may possibly be found rather difficult by immature students, but by suitable omissions no trouble would be experienced. Not only the author but the publishers as well are to be congratulated on their part in the production of the book.

HOWARD H. MITCHELL.

SHORTER NOTICES.

Elliptic Integrals. By HARRIS HANCOCK. New York, Wiley, 1917. 104 pp.

THIS volume is number 18 of the well-known series of Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. It was prepared in response to a request from the editors for a work of about one hundred octavo pages on elliptic integrals which should "relate almost entirely to the three well-known elliptic integrals, with tables and examples showing practical applications." The monograph is confined to the Legendre-Jacobi theory and the discussion is limited almost entirely to the elliptic integrals of the first and second kinds.

After a short introduction (pages 5–8), mostly historical, there follows in Chapter I (pages 9–23) an elementary discussion of the three kinds of elliptic integrals and the Legendrian transformations. The Jacobi elliptic functions are treated in Chapter II (pages 24–40). Chapter III (pages 41–64) is devoted to elliptic integrals of the first kind and Chapter IV (pages 65–87) to numerical computation of the elliptic integrals of the first and second kinds and to Landen's transformations. Several miscellaneous examples and problems are given in Chapter V (pages 88–91). In the sixth and last chapter (pages 92–101) are three five-place tables as follows: Table I, the complete integrals of the first and second kinds, page 93; Table II, elliptic integrals of the first kind, pages 94–97; Table III, elliptic integrals of the second kind, pages 99–101.

A few misprints should be mentioned: On page 6, line 4, "*Application*" is printed for "*Applications*"; on page 12, line 6 below, "within" is printed for "in"; on page 24, second exposed line, supply lower limit of integration 0, and in the third line below this replace "written" by "wrote"; at the middle of page 26, supply lower limit of integration 0; on page 30, line 4, replace "real" by "pure imaginary"; on page 32, line 3 below the figure, "rectangles" is preferable to "parallelograms."

There is lack of consistency in the use of a_0 on page 10. It gives a jolt to the reviewer to read (page 20, lines 6 and 5 below) "F_1 increases from $\frac{1}{2}\pi$ to logarithmic infinity." There is no difference between the infinities approached by $\log x$ and x itself; the difference is in the way in which this infinity is approached. It is not clear how one may "observe" the transcendental nature of K and K' by considering the series exhibited on page 26. The grammatical connections in the author's sentences are not always felicitous, as one may verify (for examples) by noticing the sentence beginning near the foot of page 59 and that beginning near the foot of page 76.

We have now done our worst in the criticism of this monograph. It has many merits to commend itself to our interest. It affords in small compass an introduction to certain portions of the theory of elliptic integrals and functions, portions which in this concise exposition will be useful to a considerable number of individuals. It certainly covers very well just the ground which the editors asked the author to cover.

Perhaps it is never proper for a reviewer to quarrel with the author for not having written a different book from the one which he did write, especially when he has carried out a request from the editors of a series. But, in the present instance, one can hardly avoid raising the question as to which is likely to be more useful now, a hundred-page monograph on elliptic integrals such as the one under consideration or a like monograph on elliptic functions approaching them from the fascinating function-theoretic points of view. There seems little room to doubt that the latter would prove of more service and have a wider distribution. It is to be hoped, therefore, that the appearance of the monograph now before us will not prevent the preparation and publication of a similar one on elliptic functions.

R. D. CARMICHAEL.

The Continuum and Other Types of Serial Order. By EDWARD V. HUNTINGTON. Second edition. Harvard University Press, 1917. vii + 82 pp.

FROM some cause the writer of this notice had never read the first edition of this book. It was therefore with a fresh, as well as a peculiar, pleasure that he followed the author in his remarkably beautiful and satisfying exposition of one of the highly fascinating subjects of modern mathematics. Any one else not at first hand acquainted with this work, whether or not he has been interested in the theory of the continuum and related matters, will find awaiting him here a purely intellectual delight of unusual order. It would be hard to seek out anywhere a more satisfying account of a topic in mathematics.

It should be emphasized that this exposition is useful to others as well as to mathematicians. Such in fact is one of its leading values. It deals with a subject which requires no technical knowledge of mathematics and which therefore is "peculiarly accessible to the increasing number of non-mathematical students of scientific method who wish to keep in touch with recent developments in the logic of mathematics." In the treatment before us full use has been made of this characteristic of the subject. The mathematical prerequisites for understanding it have been reduced (except in one or two illustrative examples) to a knowledge of the natural numbers 1, 2, 3, $\cdots$, and the simplest facts of elementary geometry. Numerous well-chosen examples are given to illustrate in a concrete way the abstract notions which are treated. The work is therefore accessible to all mature persons who are interested in purely intellectual matters and take pleasure in the more ideal æsthetic elements of thought.

The mathematical reader does not lose by this non-technical exposition; on the other hand, such a treatment has for him a peculiar value of its own. One has only to run over a general list of mathematical topics for a given definite period, as, for instance, the six months covered by a single recent issue of the *Revue Semestrielle*, to get a striking realization of the diversity of elements in modern mathematics and of the fact that mathematicians have not only largely grown apart from other scientific workers but that they have become, in small groups, isolated from one another and often are hardly able to speak across the chasm which separates one group from even its nearest neighbor. A great disintegrating tendency arises from this fact; and there seems to be a growing feeling that it is necessary to take steps to combat it. Probably this can be done in no way better than by having non-technical developments of connected bodies of doctrine in many modern subjects. For such a treatment the exposition in Huntington's Continuum may possibly serve as an ideal toward which to labor, but nevertheless an ideal which probably can be attained only in relatively rare cases owing to the nature of the topics themselves.

R. D. CARMICHAEL.

A Course in Mathematical Analysis: Differential Equations, being Part II of Volume II. By EDOUARD GOURSAT. Translated by EARLE RAYMOND HEDRICK and OTTO DUNKEL. Boston, Ginn and Company, 1917. viii + 300 pp.

To the translators it seemed best, for the purposes of American schools, to issue separately the two parts of the second volume of Goursat's Cours d'Analyse Mathématique; and this has been done with the approval of Professor Goursat. The treatise before us consists of the second half of the second volume.

This work is too widely known for us to give here a statement as to its contents; it is too favorably known for us to attempt an analysis of its main characteristics. The excellent translation is worthy of the original. The printing and in fact the whole mechanical make-up are appropriate to the contents.

On page 109 insert "the" between the fourth and fifth words of line four. On page 115, line 4, "Appel" is printed for "Appell." "Mécanique" is spelled incorrectly in the last line of page 151.

The sentence in lines 3–5 of page 102 the reviewer would be pleased to see put in stronger terms. Concerning the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y + a_{n+1} = 0$$

it is said: "It may, however, happen that a point α is a singular point for some of the coefficients a_i without being a singular point for all the integrals." Here "all the integrals" might be replaced by "any of the integrals," as one sees from the equation

$$y'' - \frac{2 \sin x}{\sin x - \cos x} y' + \frac{\sin x + \cos x}{\sin x - \cos x} y = 0,$$

of which the general solution is

$$y = c_1 e^x + c_2 \sin x.$$

Here the coefficients have poles but the general solution is an entire function. Such equations of the second order are readily formed in unlimited number by determining each one so as to have two entire functions as particular integrals, these entire functions being chosen so that their real zeros do not separate each other. This fact is an immediate corollary of Sturm's zero-separation theorem.

R. D. CARMICHAEL.

Elementi di Aritmetica, con note storiche e numerose questioni varie per le scuole medie superiori, Parte prima: *Numeri interi—Operazioni; divisibilità, numeri primi.* (Third edition, Trimarchi, Palermo, 1916. vi + 134 pp. Fourth edition, 1918. 132 pp.) By Professor GAETANO FAZZARI, of Palermo. Price, L 1.60.

THIS arithmetic includes, as is common in European texts, much algebraic material. Thus discussion of such topics as the laws of commutation and association, and the euclidean process of finding G.C.D. appear. The fundamental operations of arithmetic are discussed both from the elementary point of view and from that of the higher mathematics.

The Hindu method of "multiplication in one line," by obtaining successively those products which contain units, tens, hundreds, · · ·, is explained Division by use of the complement, frequently a convenient method, is given, illustrated by the division of 47830219 by 68947. The process is as follows, and may be regarded as division by 100000 − 31053.

$$\begin{array}{rl}
 & 31053 \\
47830219 : & 68947 \\
\underline{186318} & \overline{693} \\
6646201 & \\
\underline{279477} & \\
9256789 & \\
\underline{93159} & \\
349948 &
\end{array}$$

Similar devices, more or less illuminating to the young student, are shown in connection with the discussion of all the fundamental operations. A prominent place is given to interesting material in elementary number theory.

For half a century Professor Fazzari has been a distinguished writer on the history of science. This interest is reflected in the well-chosen historical and philological notes which appear at the end of each chapter. The exercises in 31 pages at the end of the work are selected from ancient and modern sources to illustrate curious and interesting properties of numbers.

The fourth edition is, as the preface states, practically unchanged. On its face, this edition is extremely modern, as it is dated "Palermo, 1918" and the author's preface is dated "Palermo, September, 1918.

LOUIS C. KARPINSKI.

Lehrbuch der darstellenden Geometrie für technische Hochschulen. By EMIL MÜLLER, professor of mathematics at the technical school of Vienna. Second volume, final instalment. Leipzig, Teubner, 1916. 130–360 pp.; 141–328 figures.

THE first volume of Professor Müller's book was reviewed in the BULLETIN, volume 16, page 136, and the first instalment of the second volume in volume 20, page 258. The present part is concerned with oblique axonometry and central perspective. The former is founded on Polke's theorem, a number of alternate proofs of which are given. As in the preceding parts, the text is well supplied with historical and bibliographical foot-notes, emphasizing that the science was of slow, gradual, and international growth.

The treatment of general axonometric representation is so exhaustive as to make the book hardly suitable as a text, but it is all the more serviceable as a handbook. A full discussion of æsthetic advantages and disadvantages is included, and the method is critically compared with those met with in the earlier parts of the book. Between this subject and central perspective a chapter on oblique projection is inserted, with applications to circles, spheres, and surfaces of revolution. A goodly list of exercises follows each chapter.

The development of the principles of perspective is particularly clear and readable. It is first presented independently, then shown to be approachable also from the h, v drawings, or from axonometric ones. The mathematical standpoint is never lost sight of, although no pains are spared to bring out desired artistic effects. The purely theoretical development is followed by a description of various devices to aid in drawing, including parallel rulers, Nicholson's centrolinead, Schilling's three-bar ruler, and a number of linkages. The final chapter contains an elaborate application of the method to the representation of curves and surfaces, including arches, trusses, etc.; a few pages are given to explaining the meaning and uses of photogrammetry.

The book is provided with a full index, and a list of all the authors cited. VIRGIL SNYDER.

The Method of Least Squares with Applications (third edition). By DANA P. BARTLETT. A. D. Maclachlan, 502 Boylston St., Boston, 1915. 143 + xi pp. Price $2.25.

FOR clear, concise statement in readable form and for systematic treatment of the whole range of ordinary applications of the method, we have not found the equal of this text by Professor Bartlett, of the Massachusetts Institute of Technology. These qualities have led the reviewer to adopt it for use covering this phase of the course with his students in "the mathematics of statistics."

The author has avoided making provision for special lines of work in which the method is applied, and so the instructor is free in the development when such special applications are needed. A well chosen example with complete solution follows every important development of the theory, but the 155 well graded examples are arranged like the punctuation marks in a certain school-boy's composition, which had none in the body of it but an ample supply of all sorts that might be needed at the close with the instruction "Put them in where you may wish."

The appendix gives the elements of the theory of probability, a bibliography, and three tables useful in applying the method. Throughout the text, references are made to more extended treatments in standard texts. CHARLES C. GROVE.

Text-Book of Mechanics. Vol. VI. *Thermodynamics*. By LOUIS A. MARTIN, JR. New York, Wiley, 1916.

THIS book, which is the sixth volume of a series of texts on mechanics, is a remarkably compact and comprehensive theoretical treatment of the mechanics of gases and vapors.

The first part deals with the laws of ideal gases, including a very thorough discussion of the fundamental equation $pv = RT$ and its graphical representation on the pv plane, as well as the representation of polytropic curves on log-log paper.

The first of the applications which follow is a brief chapter on compressors. The various cycles of hot air and internal combustion engines are then analyzed, although in view of the increasing importance of the latter a great deal more space might well have been devoted to a discussion of the Otto and Diesel cycles, especially in relation to the conditions under which their efficiencies approach a maximum.

This first part then concludes with a chapter on the two fundamental laws of thermodynamics, namely the law of the conservation of energy and of the flow of heat, including a discussion of Carnot's principle and Kelvin's absolute scale of temperature, and lastly a chapter on entropy and temperature-entropy diagrams.

The second part on vapors follows practically the same order of treatment as that on gases. The general properties of vapors are first discussed, followed by the properties peculiar to dry, wet, and superheated vapors. This is followed in order by chapters on the entropy of vapors, changes of state of vapors, and vapor cycles.

The third part is essentially a supplement on the flow of fluids, including the effect of throttling, the use of Venturi meters, etc., while the last chapter takes up very briefly the fundamental differential equations of thermodynamics.

An important feature of the book is a collection of 382 numerical problems.

The book is well printed and illustrated, and carefully written, and altogether is an admirable elementary presentation of the theoretical side of the subject. This subject is of such importance, especially at the present time, that an up-to-date treatment like that under discussion is a valuable addition to technical literature. As mentioned above, however, a more extended discussion of the practical applications of the subject would have added materially to the interest and usefulness of the book.

S. E. SLOCUM.

The Elements of Surveying and Geodesy. By W. C. Popplewell. New York, Longmans, Green and Company, 1915. 244 pp. Price $2.25.

This book contains the usual descriptions of surveying instruments and their uses, chapters on geodetic astronomy, earthwork calculations, railway curves, underground and hydrographic surveying. Distinctively an English text, it contains no matter pertaining to public land surveys in the United States. The exposition is clear, detailed, and accurate. The mathematical nature of the subject is kept in the foreground. A brief course, as the title indicates, it covers the field in a surprising manner.

The book contains five pages of four-place tables. There are no collections of trigonometric or other formulas. For field work it would need to be supplemented by a separate book of tables. There are no exercises.

W. V. Lovitt.

NOTES.

The third annual meeting of the Mathematical Association of America was held at the University of Chicago on Thursday and Friday, December 27–28, 1917. At the opening session on Thursday morning papers were presented by A. F. Frumveller: "The graph of $f(x)$ in line coordinates for complex numbers;" F. H. Hodge: "On the generalization of the witch and the cissoid;" W. H. Bussey: "Fermat's method of infinite descent;" C. N. Moore: "On the disciplinary and applied values of mathematical study;" E. J. Moulton: "On the content of a second course in the calculus." Thursday afternoon was devoted to a consideration of the teaching of descriptive geometry as a college subject, opening with an address by W. H. Roever, followed by extended discussion. At the meeting of the Council important matters of policy and procedure were taken up. The evening was dedicated to a joint dinner with the Chicago Section of the American Mathematical Society. On Friday morning reports were presented by the committees on mathematical requirements, libraries, and mathematical dictionary. General discussion followed each report. In the afternoon a joint session was held with the Chicago Section, after which the annual business meeting of the Association and the election of officers took place.

AT the meeting of the National academy of sciences, at Philadelphia, November 20–21, papers were read by EDWARD KASNER: "Geometric aspects of the theory of heat," and O. E. GLENN: "Invariants which are functions of parameters of the transformation."

AT the Pittsburgh meeting of the American association for the advancement of science, the retiring vice-presidential address before Section A was by Professor L. P. EISENHART on "The kinematical generation of surfaces."

THE regular summer meeting of the Swiss mathematical society was held at Zurich on September 11, 1917. The programme included papers by DAVID HILBERT: "Axiomatic thought;" C. CARATHEODORY: "Concerning the geometric treatment of extrema in double integrals;" and ARNOLD EMCH: "Concerning plane curves with certain properties."

AT the meeting of the London mathematical society on December 5 the following papers were read: by R. L. HIPPISLEY: "A new method of describing a three bar curve;" by OSCAR HOPPE: "Proof of the primality of $n = (10^{19} - 1)/9$;" by G. H. HARDY and E. J. LITTLEWOOD: "New Tauberian theorems;" by C. V. H. RAO: "The curves which lie on the quartic surface in space of four dimensions and the corresponding curves on the cubic surface and the quartic with a double conic;" by W. H. YOUNG: "The connection between Legendre series and Fourier series," and "Series of Bessel functions."

AT the meeting of the Edinburgh mathematical society on December 14 papers were read by C. TWEEDIE: "Nicole's contribution to the foundation of the calculus of finite differences;" by E. T. WHITTAKER: "An asymptotic representation of the exponential function."

THE following American doctorates with mathematics as a major subject should be added to the list published in the December, 1917, BULLETIN: J. D. BARTER, California, "A contribution to the theory of vector functions;" SISTER M. GERVAS, Catholic University of America, "On the cardioids fulfilling assigned conditions;" W. G. HUBERT, New York University, "Sextic curves with two triple points;" J. N. RICE, Catholic University of America, "On the inscribed and circumscribed triangles of the plane rational quartic curve."

THE Paris academy of sciences has awarded the Bordin prize of 3,000 francs to M. GASTON JULIA; the Francoeur prize of 1,000 francs to M. HENRI VILLAT for his publications on hydrodynamics; the Montyon prize of 700 francs to M. RENÉ DE SAUSSURE for his work in mechanics; and the Poncelet prize of 200 francs to M. JULES ANDRADE for work in applied mechanics.

PROFESSOR C. A. EPPERSON, of the Kirksville, Mo., Normal School has been commissioned first lieutenant, coast artillery reserve corps, Battery E, 61st Artillery.

PROFESSOR WILLIAM MARSHALL, of Purdue University, has entered the national food administration service and has been assigned to duty as controller and chief statistician of the international sugar committee.

PROFESSOR J. N. VAN DER VRIES, of the University of Kansas, has received a year's leave of absence to engage in the work of the field division of the U. S. chamber of commerce.

REV. A. S. HAWKESWORTH has been appointed mathematician on the staff of the chief of ordnance of the Navy Department.

AT Syracuse University associate professor F. F. DECKER has been promoted to a full professorship of mathematics. Dr. J. L. JONES has been promoted from an instructorship to an assistant professorship of mathematics.

BOOK CATALOGUES:—Galloway and Porter, Cambridge, England, catalogue 89, 50 titles in mathematics and physics.—Francis Edwards, 83 High Street, London, catalogue 377, 18 titles in mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 169, 543 titles in mathematics, physics, and engineering.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Di Dia (G.). L'omografia e l'involuzione nelle forme fondamentali di 1a specie e la dualità nella geometria proiettiva. Bologna, tip. Cuppini, 1917. L. 1.05

Ginn and Company. Mathematical portraits and pages. Boston, Ginn, 1916. 19 pp.

Henderson (A.). The twenty-seven lines upon the cubic surface. 2d edition. (Diss.) Chicago, 1915. 8vo. 6+100 pp. 12 plates in pocket.

Pérez (A.). Disquisiciones trigonométricas. Buenos Aires, Imprenta Coni Hnos, 1915.

——. Investigaciones matemáticas curiosas o útiles pero poco conocidas. Buenos Aires, 1916.

Picard (E.). Les sciences mathématiques en France depuis un demi-siècle. Paris, Gauthier-Villars, 1917. 8vo. 26 pp. Fr. 2.00

Rey Pastor (J.). Elementos de análisis algebraico. Madrid, 1917.

Wargny (C.). Reglas fundamentales de differenciación. Valparaiso, Imprenta "Barcelona," 1916. 176 pp.

II. ELEMENTARY MATHEMATICS.

Arnold (E. E.). See Durell (F.).

Dixit-Dominus (G.). Teorica grafica dimostrativa sulla trisezione degli angoli piani. Palermo, tip. Fratelli Vena, 1917. 16mo. 16 pp. con 6 tavole.

Durell (F.) and Arnold (E. E.). Plane and solid geometry. New York, Merrill, 1917. 8vo. 503 pp.

——. Solid geometry. New York, Merrill, 1917. 8vo. Pp. 18+311–499.

Fazzari (G.). Elementi di aritmetica con note storiche e numerose questione varie per le scuole medie superiori. Parte 1a: Numeri interi, operazioni, divisibilità, numeri primi. 4a edizione. Palermo, Trimarchi, 1918. 132 pp. L. 1.70

Fidencio de Alzáa (—.). Nociones de geometria plana. 1a parte, con arreglo a las ultimas programas de los colegios nacionales y escuelas normales de la República Argentina. Buenos Aires, Libreria de A. Garcia Santos, 1917.

Garzón (J. M.). Tratado de trigonometria elemental. 2a edición. Buenos Aires, 1916. 268 pp.

Harris (A. v. S.) and Waldo (L. McL.). Number games for primary grades. Chicago, Beckley-Cardy Company, 1917. 123 pp. $0.60

Miller (A. A.). Circleometry. Des Moines, The Homestead Company, 1916. 16 pp. $0.60

Thomson (J. B.). The art of teaching arithmetic. A book for class teachers. New York, Longmans, 1917. 8vo. 8+295 pp. $1.35

Waldo (L. McL.). See Harris (A. v. S.).

Whited (K.). Trisection of an angle. The problem solved geometrically with other problems. Redmond, Ore., K. Whited, 1917. Royal 8vo. 14 pp. Paper. $1.00

III. APPLIED MATHEMATICS.

BIANCHI (E.). Tavole astronomiche di latitudine e longitudine (Ministero della guerra: direzione generale d'aeronautica, istituto centrale aeronautico). Roma, tip. R. Accademia dei Lincei, 1917. 8vo. 16+24 pp.

BRADLEY (H. C.). See KENISON (E.).

CANDIANI (G.). Gli elementi per lo studio della resistenza al moto dei solidi nei fluidi: comunicazione letta al IX congresso della Società italiana per il progresso delle scienze, Milano, 4 aprile, 1917. Padova, Società coop. tipografica, 1917. 16mo. 15 pp.

CLAVARINO (A.). Tavole per il calcolo degli elementi riguardanti una traiettoria qualunque. Vicenza, tip. G. Rumor, 1917. 24mo. 35 pp.

CREPALDI (E. L.). Lezioni di meccanica dei motori a scoppio, svolte ai corsi allievi ufficiali automobilisti e volontari automobilisti in Padova. 3a edizione. Padova, La Litotipo, 1917. 8vo. 287 pp. L. 5.50

DUFOUR (P. T.). Nouveau procédé permettant d'obtenir les perspectives-reliefs des formes géographiques représentées sur les cartes hypsométriques. Paris, Delagrave, 1917. 4to.

EFFEMERIDI astronomiche per il 1917 (Ministero della guerra: direzione generale d'aeronautica, istituto centrale aeronautico). Roma, tip. R. Accademia dei Lincei, 1917. 8vo. 41 pp.

ELEMENTI geodetici dei punti contenuti nei fogli 130 e 131 (Orvieto, Foligno) della carta d'Italia (Istituto geografico militare). Firenze, tip. Barbèra, di Alfani e Venturi, 1917. 4to. 70+85 pp. con 2 tavole.

GOFFI (C.). Tecnica moderna degli acciai: manuale per gli operai aggiustatori meccanici. (Manuali Hoepli.) Milano, Hoepli, 1917. 24mo. 19+260 pp. con tavole. L. 4.50

GRASSO (C.). Elementi di tecnologia, comprendente la materia dei programmi ministeriali per la 2a e 3a classe industriale, sezione meccanici elettricisti. 2 volumi. Napoli, casa ed. Elpis (S. Morano), 1917. 16mo. 170+205 pp. L. 5.75

HAMMICK (D. L.). See PERRIN (J.).

HEATH (T. E.). The distances, absolute magnitudes, and spectra of 734 stars. Arranged for use with ordinary star maps. Tenby, Miss Crealock, 1917. 4 + 52 pp. 2s.

KENISON (E.) and BRADLEY (H. C.). Descriptive geometry. New York, Macmillan, 1917. 10+287 pp. Cloth. $2.00

LEPPER (G. H.). From nebula to nebula or the dynamics of the heavens. Third edition rewritten and enlarged. Pittsburgh, G. H. Lepper, 1917. 363 pp. $3.50

MAGGI (G. A.). Elementi di statica e teoria dei vettori applicati. Pisa, E. Spoerri (s. tip.), 1917. 8vo. 4+175 pp.

MASSERO (F.). Manuale pratico per l'aggiustatore meccanico. (Manuali Hoepli.) Milano, Hoepli, 1917. 24mo. 12+263 pp. L. 4.50

PERRIN (J.). Atoms. Translated by D. L. Hammick. New York, Van Nostrand, 1916. 16+211 pp. $2.50

ZALDARI (P.). Annuities and amortization tables. New York, Bankers Encyclopedia Company, 1917. 350 pp. Morocco. $10.00

L'' RARY

THE TWENTY-FOURTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fourth annual meeting of the Society was held in New York City on Thursday and Friday, December 27–28, 1917, extending through two sessions on Thursday and a morning session on Friday. The attendance included the following forty-six members:

President R. J. Aley, Professor Joseph Bowden, Dr. T. H. Brown, Professor W. B. Carver, Professor F. N. Cole, Dr. G. M. Conwell, Dr. J. V. DePorte, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. G. M. Green, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Dr. T. R. Hollcroft, Professor L. A. Howland, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor P. H. Linehan, Professor C. R. MacInnes, Professor Helen A. Merrill, Professor R. L. Moore, Professor Frank Morley, Dr. G. W. Mullins, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor Clara E. Smith, Professor D. E. Smith, Professor Sarah E. Smith, Professor H. D. Thompson, Mr. J. N. Vedder, Professor C. W. Watkeys, Mr. H. E. Webb, Mr. R. A. Wetzel, Professor E. E. Whitford, Professor Ruth G. Wood, Professor J. W. Young.

At the opening session Professor R. G. D. Richardson took the chair. Professor J. W. Young presided at the following sessions. The Council announced the election of the following persons to membership in the Society: Dr. J. W. Campbell, Wesley College, Winnipeg, Canada; Dr. Mary F. Curtis, Western Reserve University; Mr. C. H. Parsons, Columbia University; Mr. J. B. Rosenbach, University of New Mexico; Mr. H. M. Terrill, Columbia University. Five applications for membership in the Society were received.

Committees were appointed to arrange for the summer meeting at Dartmouth College in 1918 and for the summer meeting and colloquium at the University of Chicago in 1919.

The total membership of the Society is now 735, including 77 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 351; the number of papers read was 167. The number of members attending at least one meeting during the year was 217. At the annual election 116 votes were cast. The Treasurer's report shows a balance of $9,762.98, including the life-membership fund of $6,333.13. Sales of the Society's publications during the year amounted to $1,474.19. The Library now contains 5,475 volumes, excluding unbound dissertations.

At the annual election, which closed on Friday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor J. L. COOLIDGE, Professor D. R. CURTISS.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1920,

Professor R. C. ARCHIBALD,	Professor D. N. LEHMER,
Professor DUNHAM JACKSON,	Professor J. B. SHAW.

The following papers were read at the annual meeting:

(1) Professor F. L. HITCHCOCK: "The coincident points of two algebraic transformations."

(2) Professor W B. CARVER: "The conditions for the failure of the Clifford chain."

(3) Professor C. J. KEYSER: "The rôle of the concept of infinity in the work of Lucretius."

(4) Professor C. J. KEYSER: "Concerning the number of possible interpretations of any system of postulates."

(5) Dr. W. H. WILSON: "Systems of functional equations which define hyperbolic sine, hyperbolic cosine, sine, and cosine uniquely."

(6) Dr. C. H. FORSYTH: "Tangential interpolation of ordinates among areas."

(7) Professor W. B. FITE: "Concerning the zeros of the solutions of certain differential equations."

(8) Professor R. L. MOORE: "Concerning a set of postulates for plane analysis situs."

(9) Dr. C. A. FISCHER: "Integral equations involving Stieltjes integrals."

(10) Professor O. E. GLENN: "Preliminary report on a new treatment of theorems of finiteness."

(11) Professor C. L. E. MOORE: "Rotations in hyperspace."

(12) Dr. G. M. GREEN: "Memoir on the general theory of surfaces and rectilinear congruences."

(13) Dr. J. F. RITT: "On the iteration of rational functions."

(14) Dr. OLIVE C. HAZLETT: "On rational integral invariants and covariants of the general linear algebra."

(15) Professor ANNA J. PELL: "Systems of linear equations."

(16) Dr. NORBERT WIENER: "Internal isomorphisms of complex algebra."

(17) Dr. T. R. HOLLCROFT: "A classification of general (2, 3) point correspondences between two planes."

(18) Professor W. F. OSGOOD: "Singular points of analytic transformations."

(19) Dr. M. T. HU: "Linear integro-differential equations with a boundary condition."

(20) Professor FRANK MORLEY: "Some general projective invariants of the algebraic planar curve."

The papers of Professor Hitchcock and Dr. Wilson were communicated to the Society through Professor C. L. E. Moore. Dr. Wiener was introduced by Professor Huntington. Dr. Hu's paper was communicated through Professor Bôcher. In the absence of the authors the papers of Professor Hitchcock, Dr. Wilson, Dr. Forsyth, Professor Glenn, Professor C. L. E. Moore, Professor Osgood, and Dr. Hu were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Let $x_1, x_2, \cdots, x_n$ be a set of n independent variables. For convenience of language we may say that these variables are the homogeneous coordinates of a point in space of $n - 1$

dimensions. Let $P_1, P_2, \cdots, P_n$ and $Q_1, Q_2, \cdots, Q_n$ be two sets of polynomials homogeneous in the variables x and of degrees p and q respectively, defining a pair of transformed points P and Q. There always exist one or more points x such that P and Q coincide. This is the same as saying that the two-row determinants of the matrix

$$\begin{matrix} P_1, P_2, \cdots, P_n \\ Q_1, Q_2, \cdots, Q_n \end{matrix}$$

vanish simultaneously for certain properly chosen sets of values of the variables $x_1, x_2, \cdots, x_n$ Professor Hitchcock shows that the number of such points is in general $p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + q^{n-1}$.

2. Clifford discovered a chainwise extension of the notion of the circumcircle of 3 lines, and showed* that n lines determine a circle when n is odd and a point when n is even. Professor Morley has given† analytic expressions for the Clifford circles and points in conjugate complex coordinates, and has shown that the Clifford circle becomes a straight line when a certain determinant vanishes. In the present paper Professor Carver shows that the Clifford chain fails when, and only when, the n lines are tangents to certain metric curves. These curves are rational, and of a type the dual (projectively) of the Jonquière curves.

3. Professor Keyser shows that the scheme of thought in the De Rerum Natura of Lucretius is dominated by the concept of infinity; that the author's conception of an infinite multitude accords with the current definition of the term; that the same is true of his notion of an infinite magnitude; that the infinites of Lucretius are not variables but are, like those of Cantor, static affairs; that, on the other hand, the Lucretian infinites are not composed of abstract things (points of space and pure numbers) but of concrete things (material particles, for example); that a Lucretian infinite multitude is always of the denumerable variety; and that the poet made repeated use of the characteristic whole-part property of infinite multitudes.

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* "Synthetic proof of Miquel's theorem," Clifford's Mathematical Papers, p. 38 (1870).

† "On the metric geometry of the plane n-line," *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 97.

4. The main purpose of this note by Professor Keyser is to show that any postulate system involving at least one undefined term denoting an element as distinguished from a relation admits of any given infinite number of different interpretations.

5. In his Lehrbuch der Funktionentheorie, second edition, page 582, Professor Osgood has given a determination of the functions $\sin x$ and $\cos x$ on the basis of their addition theorems. In the present paper Dr. Wilson determines the functions $f(x) = \sin ax$ and $g(x) = \cos ax$ as the general continuous solutions of the simultaneous equations

$$g(x - y) = g(x)g(y) + f(x)f(y),$$
$$f(x - y) = f(x)g(y) - g(x)f(y),$$

where x and y are independent real variables and $f(x) \not\equiv 0$. It is furthermore shown that if x and y are not real and if $x = u + v\sqrt{-1}$, u and v real, then the general solutions continuous over the finite complex x-plane are

$$g(x) = \cos(au + bv), \qquad f(x) = \sin(au + bv),$$

the general solutions continuous along any line in the finite complex x-plane not parallel to the axis of reals are

$$g(x) = \cos bv, \qquad f(x) = \sin bv,$$

and the general solutions continuous along any line in the finite complex x-plane not parallel to the axis of imaginaries are

$$g(x) = \cos au, \qquad f(x) = \sin au,$$

where a and b are arbitrary constants. A similar determination of the functions $g(x) = \cosh ax$ and $f(x) = \sinh ax$ is made from the simultaneous equations

$$g(x - y) = g(x)g(y) - f(x)f(y),$$
$$f(x - y) = f(x)g(y) - g(x)f(y).$$

6. Dr. Forsyth modifies an interpolation formula used to interpolate ordinates when the data consist of areas so that in interpolating several ordinates in each of several successive intervals the discontinuities ordinarily occurring at the points

of intersection of any two consecutive interpolation curves are practically eliminated by requiring the slopes of these curves to be the same at the intersections.

7. In this paper Professor Fite generalizes the results of Kneser published in volume 42 of the *Mathematische Annalen* concerning certain linear differential equations whose solutions change sign an infinite number of times as the independent variable increases indefinitely through real values. He also establishes the existence of a minimum length for an interval within which all the functions which form a solution of a system of linear equations can vanish.

8. In his paper "On the foundations of plane analysis situs,"[*] Professor Moore exhibited three systems of axioms, Σ_1, Σ_2, and Σ_3. He proposes to show that every space that satisfies Σ_1 or Σ_2 is a number plane in the sense that it is in one-to-one continuous correspondence with euclidean space of two dimensions.

9. In this paper Dr. Fischer extends the definition of the Stieltjes integral $\int f(x)d\alpha(x)$ so that it is determined uniquely whenever both f and α have finite variation. It is then proved that when $f(x)$ and $K(x, y)$ have finite variation, and when the variation of $K(x, y)$ considered as a function of y alone is bounded, the equation

$$f(x) = u(x) + \lambda \int_a^b u(y)d_yK(x, y)$$

has a unique, finite solution, for sufficiently small values of λ. It is also proved that this equation can under certain conditions be put into the form

$$g(x) = u(x) + \lambda \int_a^b L(x, y)du(y),$$

and that this last equation has a solution for sufficiently small values of λ, when it is merely assumed that $g(x)$ has finite variation, that $L(x, y)$ is finite, and that when it is considered as a function of x alone its variation is bounded. Both the method of successive substitutions and that of reciprocal functions are employed.

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[*] *Transactions*, vol. 17, no. 2 (April, 1916), p. 131.

10. The rationale of the method of Professor Glenn's paper is based upon a property, which he has established, that all concomitants of any one of several designated infinite systems are expressible as polynomials in certain elemental concomitants belonging to specified domains. Hilbert's theorem then applies.

The theory is adaptable to formal concomitants of a quantic to a prime modulus, concerning which further investigation is in progress.

11. Professor Moore first shows that a complex 2-vector in a space of $2m$ dimensions can always be resolved into the sum of m mutually completely perpendicular plane vectors. This resolution is not unique in case certain relations exist between products of these vectors. In four dimensions the resolution is not unique if the 2-vector is self-complementary. From these facts it is shown that a rotation in $2m$ dimensions can be resolved into rotations parallel to m mutually completely perpendicular planes. This resolution is not unique if the rates of rotation in two or more of the planes are equal.

12. In Dr. Green's paper certain geometric concepts introduced by him in previous communications to the Society are applied in the study of a curved surface and of certain associated rectilinear congruences. A line l in the tangent plane of the surface has related to it a line l' passing through the corresponding point of the surface, the lines l and l' being reciprocal polars with respect to the osculating quadric of that point. The lines l and l' are in the general relation R with respect to the asymptotic net of the surface, according to a definition introduced by Dr. Green in other papers.* A congruence Γ composed of lines l determines a reciprocal congruence Γ' of lines l'. By further specializing the relation between the congruences Γ and Γ' various important congruences may be defined which are uniquely determined by the surface. Such specialization of the relation between Γ and Γ' seems to be very fruitful both in the discovery of new congruences and in the characterization of known ones. Among these are the directrix congruences of Wilczynski, a pair of reciprocal congruences which reduce to Sullivan's

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* Cf. in particular a recent note in the *Proceedings of the National Academy of Sciences*, vol. 3, no. 10 (Oct., 1917).

scroll directrix congruences when the surface is ruled, and the canonical congruences and pseudo-normal congruence introduced by Dr. Green at the summer meeting of 1916 and described in the note already cited.

A congruence Γ is said to be harmonic to a surface S if its developables correspond to a conjugate net on S, and a congruence Γ' is said to be conjugate to S if its developables correspond to a conjugate net on S. Characterizations are given for such congruences, and various questions are studied in connection with them. In particular, the characterization of planar nets with equal invariants is found to be closely related to the case in which the developables of the congruence Γ' are indeterminate.

The developables of a congruence Γ correspond to a net of curves on S, and the tangents to these curves are called Γ-tangents. The line l is met in its focal points by the tangents which are conjugate to the corresponding Γ-tangents, a property which has some important geometric consequences, in particular the geometric characterization of congruences harmonic to a surface.

13. Let $f_n(x)$ be the nth iterate of a rational function $f(x)$. The theorems presented by Dr. Ritt deal principally with the antecedents of any point a, that is, the points x such that, for some n, $f_n(x) = a$, and with the associates of a, the points x such that, for some n, $f_n(x) = f_n(a)$. The antecedents of a point are generally infinite in number, and the set of their limiting points contains a perfect set, which probably is an analytic curve; a similar statement holds for the associates. Perhaps the most interesting result given, part of which was found before by Professor A. A. Bennett, and which is an immediate consequence of the existence of certain meromorphic functions discovered by Poincaré, is the following:

If $f(a) = a$, and $|f'(a)| > 1$, then, given any point b, there is an antecedent of b in any neighborhood of a, however small.

It is shown also that in any neighborhood of such a point a there are an infinite number of points x such that $f_n(x) = x$ for some n.

14. Dr. Hazlett's paper proves some theorems on rational integral invariants and covariants for general linear algebras, analogous to well-known theorems in the classical invariant

theory for algebraic forms. For instance, such invariants must be homogeneous, and must possess a certain property which (by reason of the analogy with the classical invariant theory for algebraic forms) is called isobarism. Similarly for covariants. By the aid of these and other fundamental properties, it follows at once that the rational integral invariants are finite. In the cases $n = 1, 2$ all algebraic invariants and covariants are invariants and covariants respectively of the characteristic determinants. This is the converse of a theorem proved in a previous paper.

15. By means of an orthogonal system of linear forms and linear differential forms, Mrs. Pell shows that the theory of the system of linear equations

$$x_i = c_i + \sum_k a_{ik} x_k + \sum_\alpha \int_a^b \frac{da_i^{(\alpha)}(t)\,du^{(\alpha)}(t)}{da_0^{(\alpha)}(t)},$$

$$u^{(\alpha)}(s) = f(s) + \sum_k a_k^{(\alpha)}(s) x_k + \sum_\beta \int_a^b \frac{dA^{(\alpha,\beta)}(s,t)\,du^{(\beta)}(t)}{da_0^{(\beta)}(t)},$$

where $\{x_i\}$ and $\{u^{(\alpha)}(s)\}$ are unknown, is reduced to the theory of the linear equations

$$x_i = d_i + \sum_k b_{ik} x_k.$$

A special case is the theory of mixed linear integral equations. Existence and properties of solutions of a symmetric system are obtained.

16. An internal isomorphism of a system of elements and operations thereon is a one-to-one correspondence between the elements of the system and themselves which transforms operations belonging to the system into operations belonging to the system. Two such isomorphisms are considered as the same if they induce the same transformation of the operations of the system, independently of what they do to the elements. Dr. Wiener shows that the internal isomorphisms of the system in which the elements are the complex numbers and the operations are such that a polynomial in the operations and the operands with integral coefficients is zero, are all established by linear transformations.

17. A general (2, 3) point correspondence between two

planes (x) and (x') is said to exist when to any point of (x') correspond three points of (x) and to any point of (x), two points of (x') such that the correspondence in neither plane is involutorial. Such a correspondence is defined by pairs of algebraic equations in the coordinates (x_1, x_2, x_3) and (x_1', x_2', x_3') whose loci are systems of curves in the planes (x) and (x') intersecting in three and two points respectively. Each such system gives rise to a transformation relating curves of one plane to curves of the other and determines in each plane a locus of points two of whose image points in the other plane coincide, and likewise the loci of these coincident image points. The system is said to be independent if it cannot be reduced to another by any series of birational transformations. In all, Dr. Hollcroft finds twelve such independent systems and finally proves that any system defining a general (2, 3) point correspondence between two planes is birationally equivalent to some one of these twelve types.

18. A preliminary notice of Professor Osgood's paper appeared in the BULLETIN for June, 1917. The paper will appear in full in the *Transactions*.

19. Dr. Hu studies a linear integro-differential equation of a very general type with a linear boundary condition, and introduces such conceptions as adjoint systems and Green's functions.

20. After a suggestion for obtaining the discriminant, Professor Morley considers the degree of the conditions that a line meet the curve thrice at one point and twice at another, or twice at three points.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE fortieth meeting of the Chicago Section was held at the University of Chicago on December 28–29 in conjunction with the third annual meeting of the Mathematical Association of America. The meeting opened with a dinner of the two organizations on Thursday evening, December 27, at which seventy-three persons were present. The program for the joint session on Friday afternoon consisted of the retiring address by the chairman of the section, Professor W. B. Ford, entitled "A conspectus of the modern theory of divergent series," and of a paper by Professor L. D. Ames, "On a definition of the real number system by means of infinite decimals."

The attendance at the different sessions, which continued until Saturday afternoon, included about eighty-five persons, among whom were the following fifty-nine members of the Society:

Professor L. D. Ames, Professor R. C. Archibald, Professor G. A. Bliss, Professor P. P. Boyd, Professor H. T. Burgess, Professor W. H. Bussey, Professor W. D. Cairns, Professor Florian Cajori, Professor D. F. Campbell, Professor R. D. Carmichael, Professor C. E. Comstock, Professor A. R. Crathorne, Mr. G. H. Cresse, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Professor E. L. Dodd, Professor Arnold Dresden, Professor Otto Dunkel, Professor Arnold Emch, Professor L. C. Emmons, Professor H. J. Ettlinger, Professor W. B. Ford, Professor Harris Hancock, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor F. H. Hodge, Professor E. V. Huntington, Professor A. M. Kenyon, Dr. H. R. Kingston, Professor Kurt Laves, Dr. E. B. Lytle, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor G. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor E. H. Moore, Professor C. C. Morris, Professor E. J. Moulton, Professor G. W. Myers, Professor H. B. Phillips, Professor L. C. Plant, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Dr. G. W. Smith, Professor E. J. Townsend, Professor H. W. Tyler, Professor J. N. Van der Vries, Professor L. G. Weld, Professor E. J. Wilczynski, Professor A. E. Young, Professor J. W. A. Young.

At the business meeting on Saturday forenoon the following officers were elected: chairman, Professor G. A. Bliss; secretary, Professor Arnold Dresden; third member of programme committee, Dr. A. J. Kempner. The section decided to elect its officers for terms of two years, the new practice to start with the officers elected at this meeting. A committee was appointed to consider the advisability of holding a symposium at the next spring meeting and to make the necessary arrangements in case the decision is favorable.

The sessions of Friday forenoon and Saturday forenoon were presided over by Professor Ford, chairman of the Section; the joint session of Friday afternoon by the president of the Society, Professor L. E. Dickson, and the session of Saturday afternoon by Professor G. A. Bliss, chairman-elect of the Section.

The following papers were presented at this meeting:

(1) Dr. G. E. Wahlin: "A study of the principal units of an algebraic domain $\mathfrak{k}(\mathfrak{p}, \mathfrak{a})$."

(2) Dr. C. H. Forsyth: "Supernormal curves."

(3) Professor E. L. Dodd: "A comparison of certain functions of measurement, assuming the Gaussian probability law which provides for a constant error."

(4) Dr. E. W. Chittenden: "On the limit functions of sequences of continuous functions converging relatively uniformly."

(5) Dr. A. R. Schweitzer: "On the iterative functional compositions of index $(n+1, k)$, $(n, k = 1, 2, \cdots)$ and associated functional equations."

(6) Dr. A. R. Schweitzer: "On a description of mathematics."

(7) Professor Arnold Emch: "On the theory of isotropic curves."

(8) Professor H. B. Phillips: "The relation of cubical determinants to the double multiplication of Gibbs."

(9) Professors H. B. Phillips and C. L. E. Moore: "The dyadics occurring in a projective point space of three dimensions."

(10) Professor R. D. Carmichael: "On sequences of integers defined by recurrence relations."

(11) Professor C. N. Moore: "On the convergence of the developments in Bessel's functions."

(12) Professor G. A. Miller: "Sets of independent generators of a substitution group."

(13) Professor J. B. Shaw: "Infinite linear associative algebras."

(14) Professor H. T. Burgess: "Note on the reduction of a pencil of singular quadratic forms."

(15) Dr. H. R. Kingston: "On isothermal nets of curves."

(16) Dr. A. J. Kempner: "On polynomials and their residue systems."

(17) Dr. A. L. Nelson: "Note on seminvariants of systems of partial differential equations."

(18) Professor G. A. Bliss: "Solutions of differential equations as functions of the constants of integration."

(19) Professor H. J. Ettlinger: "On the non-self-adjoint linear system of the second order."

(20) Professor R. D. Carmichael: "Repeated solutions of a certain class of linear functional equations."

The papers of Drs. Wahlin, Forsyth, Chittenden, Kingston, Kempner, and Nelson were read by title. Abstracts, numbered to correspond to the titles in the list above, follow below.

1. In a recent number of the Bulletin Dr. Wahlin published an article in which he gave a study of the group of residues of the principal units of $\mathfrak{k}(\mathfrak{p}, \mathfrak{a})$ with respect to a certain modulus.

The present paper is a closer study of the principal units along the same lines. The principal units are classified into regular and irregular units, the classification being dependent upon a certain congruence property. Then the smallest number of irregular units that can occur in the base of the group of residues is determined.

2. Perhaps the most useful curve is the Gaussian or probability curve whose equation is $y = y_0e^{-(x^2/k)}$ or, written in the complete form,

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2},$$

where N is the total area of the curve and σ is the standard deviation, usually defined as the square root of the second unit moment about the mean. However, the curve is ill adapted for one of its most common uses—that of graduation, because the equation is too rigid, having essentially only one parameter. Dr. Forsyth proposes in this paper what amounts to the addition of a second parameter by allowing the exponent of x to be any even integer, the general equation becoming $y = y_0e^{-x^{2n}/k}$. The values of y_0 and k are then expressed in terms of moments which may be computed for any distribution proposed for graduation. A formula is then derived for determining the most appropriate value of n to be used for graduating a given distribution. A short table of ordinates of the curve for $n = 2$ is added to be used in graduations.

3. Professor Dodd makes a comparison of functions of measurements, postulating that the probability that the error of a measurement will be x is

$$\frac{h}{\sqrt{\pi}}e^{-h^2(x-c)^2}\,dx,$$

where c is the unknown constant error. A comparison with c not zero cannot always be directly inferred from the corresponding comparison with c equal to zero, as was shown previously.* The following extension is, however, immediate: The arithmetic mean is superior to any other weighted mean and to the median, from the standpoint of error-risk, even with c not zero. But if the constant error c is present, then the more numerous the measurements, the more nearly certain it is that the arithmetic mean will incorporate this error c. Thus, in certain cases where an actual "hitting of the mark" is required, the making of measurements beyond a certain number is prejudicial to accuracy; although in the usual case—where "good average results" are desired—the more numerous the measurements, the more satisfactory will be the arithmetic mean, even when a constant error is present.

4. According to Professor E. H. Moore a sequence of functions f_n defined on a range $\mathfrak{P}$ converges relatively uniformly to a limit function f if there exists a scale function σ defined on $\mathfrak{P}$ such that for every small positive number e there exists an integer n_e such that for all $n \geqq n_e$

$$|f_n - f| \leqq e\,|\sigma|$$

uniformly on $\mathfrak{P}$.

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* *Monatshefte für Mathematik und Physik*, vol. 24 (1913), pp. 270, 271.

If the functions f_n are continuous on an interval $\mathfrak{P} \equiv (a, b)$, it is shown by Dr. Chittenden that the discontinuities of the limit function f relative to any perfect subset $\mathfrak{P}_0$ of $\mathfrak{P}$ cannot be dense on $\mathfrak{P}_0$ if the convergence is relatively uniform. The condition is also sufficient. From this result it is easy to show that the convergence of Fourier series is not always relatively uniform. Various consequences of this result in relation to the Baire system of classification of functions are considered.

5. Dr. Schweitzer constructs[*] the complete ordered set of iterative compositions of order k and degree $n + 1$ $(n, k = 1, 2, 3, \cdots)$, i. e., iterative compositions of k functions each of which contains not more than $n + 1$ variables. Two cases are distinguished according as the k functions all have the same number of variables or not. In the former case one obtains the complete list of compositions of order $k > 1$ and homogeneous degree, merely by assigning suffixes $1, 2, 3, \cdots$ in all possible ways to the functional symbols involved in the set of compositions when $k = 1$ and correspondingly affecting by superscripts the arguments of the new functions introduced. In the latter case the required iterative compositions are obtained from the compositions of the former case by increasing or diminishing in all possible ways the number of variables involved, the operation being subject to the notation of the functional symbols of which the variables are arguments.

In the second part of the paper the author constructs "systematically" functional equations corresponding to the preceding iterative compositions by means of the iteration, inversion, elimination, or transformation of variables. A general equation in iterative compositions with all its variables notationally distinct is defined, homogeneous or non-homogeneous with reference to order and degree. Solutions of simultaneous systems are obtained. Examples:

I. $\phi_1\{\phi(x_1, \cdots, x_{n+1}), t_1, \cdots, t_n\} = \phi\{\phi_1(x_1, t_1, \cdots, t_n), \cdots, \phi_1(x_{n+1}, t_1, \cdots, t_n)\}$. For $n = 1$ solutions of this functional equation have been given by Hankel[†] and the author.

II. $f\{\phi(x, z), f(x, y)\} = \psi(y, z)$, $f_1\{f(x, z), f(x, y)\} = \psi(y, z)$, $f_2\{\phi(x, z), f(x, y)\} = f(y, z)$.

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[*] Cf. BULLETIN, vol. 23, pp. 301, 302. In Theorem III on p. 302 the specific group of order four is 1, (14)(23), (14), (23).

[†] Theorie der Complexen Zahlen, p. 34.

III. $\phi\{\phi_1(x_{11}, \cdots, x_{1k}), \cdots, \phi_1(x_{n1}, \cdots, x_{nk})\} = \phi_1\{\phi(x_{11}, \cdots, x_{n1}), \cdots, \phi(x_{1k}, \cdots, x_{nk})\}$ $(k \leqq n, n = 2, 3, \cdots)$.

IV. $f_1\{f_1(x_{11}, \cdots, x_{1n}), \cdots, f_1(x_{n1}, \cdots, x_{nn})\} = f\{f_1(x_{11}', \cdots, x_{1n}'), \cdots, f_1(x_{n1}', \cdots, x_{nn}')\}$, where $x_{\iota k}'$ are certain linear homogeneous transformations of $x_{\iota k}$ with coefficients $m_{\iota k}$. In connection with equations of this type it is shown that if

$$f(x_{11}, \cdots, x_{1n}) = \sum_{\iota=1}^{n} a_{1\iota}x_{1\iota}$$

and

$$f_1(x_{11}, \cdots, x_{1n}) = \sum_{\iota=1}^{n} a_{1\iota}'x_{1\iota},$$

the coefficients $a_{1\iota}'$ are linear homogeneous functions of the coefficients $a_{1\iota}$ with m's as coefficients. Consequently a group of iterative functional equations of the second order is readily defined.

6. Dr. Schweitzer's definition of mathematics is as follows: mathematics is the systematic construction of clear concepts of real numerical reference, mediate or immediate. The author carefully outlines his meaning, philosophically, of the words in this definition; in particular, his interpretation of "systematic" is very largely Neo-Kantian.

7. Professor Emch's paper deals with the generation of rational circular, in particular, with rational isotropic curves. Every rational curve in a complex z'-plane may be obtained as the transformation of the real axis of a complex z-plane by a rational transformation $z' = f(z)/g(z)$, with coefficients from the domain of ordinary complex (including real) numbers. By such a transformation an algebraic curve is transformed into a curve of the same deficiency, so that rational curves are transformed into rational curves. Necessary and sufficient conditions are established for the polynomials $f(z)$ and $g(z)$, in order that the transformed curves may be circular and, in particular, isotropic. It is shown how these conditions depend upon the theorem: if a polynomial equation $g(z) = 0$, with real coefficients, has imaginary roots only, then $g(z)$ may be represented as the sum of the squares of two polynomials of the same type.

As an application, the generation of all bicircular rational quartics is given explicitly. For example, Bernoulli's lemniscate

$$(x'^2 + y'^2)^2 - 2a^2(x'^2 - y'^2) = 0$$

is represented by

$$z' = \frac{-a\sqrt{2}(z^2 - 1)}{z^2 + 2iz + 1}.$$

8. In this paper, Professor Phillips shows that a cubical determinant of the nth order is the Gibbs multiple product of n dyadics in a space of n dimensions. More generally, the multiple product of m (less than n) such dyadics is a cubical matrix of the nth order with m sheets. As a particular application, it is shown that these cubical matrices can be used instead of determinants in Hedrick's law of analogy which states that a theorem in cubical determinants corresponds to each homogeneous identity in ordinary algebra.

9. If we classify dyadics by the dimensions of their factors, there are four types of dyadics in three dimensions: the one-three and three-one, representing collineations; the one-one and three-three, representing correlations; the two-two, not in general representing a contact transformation; and the one-two, two-three, and their conjugates. Using the symbolic notation introduced by Phillips in a former paper, Professors Phillips and Moore give a treatment of the foregoing types of dyadics and the invariants obtained from them by simple and double multiplication with each other and with idemfactors. This paper will appear in the *Proceedings of the American Academy of Arts and Sciences.*

10. Special classes of recurrent sequences of integers have been investigated in detail by Lucas and others and numerous interesting theorems have resulted from such study. The purpose of the present paper by Professor Carmichael is to develop certain general properties of the general sequence of integers $u_0, u_1, u_2, \cdots$ defined uniquely in terms of the given initial numbers $u_0, u_1, \cdots, u_{k-1}$ by the recurrence relation

$$u_{x+k} + \alpha_1 u_{x+k-1} + \cdots + \alpha_k u_x = \alpha,$$

in which $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are given integers. The leading fundamental properties of such sequences are developed by elementary means and several applications of them are given. In particular, many theorems due to Fermat (among them some of his most remarkable discoveries) are readily derived from general results to which one is thus led naturally by elementary means. This association of so many results of Fermat with a single elementary theory tends to suggest the conjecture that the methods of this paper may have been employed by Fermat in some of his investigations.

11. The principal result of Professor Moore's paper is contained in the following theorem: If $f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, and in an interval $0 \leqq x \leqq c < 1$ has a first derivative which has a Lebesgue integral and which approaches a finite limit as x approaches $+0$, then the development of $f(x)$ in Bessel's functions of order $\nu (\nu \geqq 0)$ will be convergent to the value $f(+0)$ for $x = 0$, provided $\nu = 0$ or $f(+0) = 0$, and will be uniformly convergent to the value $f(x)$ in the interval $0 \leqq x \leqq c$, provided $\nu = 0$ or $f(+0) = 0$.

12. A set of λ substitutions contained in a given substitution group G is called a set of independent generators of G whenever these λ substitutions generate G, but no $\lambda - 1$ of them have this property. It is clearly always possible to select more than one set of independent generators of G whenever the order of G exceeds 2, and λ is not always the same for the various possible sets of generators of G. In particular, when G is the symmetric group of degree n it is well known that sets can be so chosen that λ may have any value from 2 to $n - 1$, but it does not seem to have been proved that λ can never exceed $n - 1$ when G is a substitution group of degree n. When G is cyclic it is easily seen that λ may have any value from one to the number of different prime factors of the order of G, but it can have no other value.

In the present paper Professor Miller proves the following theorems: If G is any substitution group of degree n with k systems of intransitivity it is always possible to generate G by $n - k$ of its substitutions. When n is written in the form

$$a_0 p^m + a_1 p^{m-1} + \cdots + a_m,$$

where m is a positive integer and each of the positive integers $a_0, a_1, \cdots, a_m$ is less than the prime number p, then the number of substitutions in each of the possible sets of independent generators in the Sylow subgroup of order p^α contained in G is $a_0m + a_1(m - 1) + \cdots + a_{m-1}$. If one of the transitive constituents of the intransitive subgroup of index p contained in a transitive group G of order p^β is abelian, the number of the independent generators of G cannot exceed one plus the number of the independent generators of this transitive constituent increased by the number of its independent generators whose order exceeds p.

13. In Professor Shaw's paper the general theorems of linear associative algebra are extended from algebras of a finite number of units to algebras with an infinite number of units. These algebras are of two kinds: one having an infinite number of idempotents, the other without idempotents and having no characteristic equations. The latter are called infinitipotent algebras. The most general linear associative algebra will then consist of one infinite or finite semi-simple subalgebra, an infinite or finite nilpotent subalgebra, and an infinitipotent subalgebra. The general theorem of finite algebras is extended to infinite algebras. Some applications are made to recent analysis along the lines of integral equations and integro-differential equations.

14. The reduction of a pencil of singular quadratic forms whose matrix $\lambda A + B$ is singular for all values of λ has been accomplished in a variety of ways. None of the schemes found in the literature enable one to compute the matrix of the reducing substitution with any degree of satisfaction.

Professor Burgess outlines in this note a simple and direct method of accomplishing the reduction, and at the same time computing the matrix of the reducing substitution.

15. In a former paper read before the Society Dr. Kingston discussed some metric properties of nets of plane curves by means of a completely integrable system of three linear partial differential equations of the second order. In particular, the condition for isothermality of the nets was found. In the present paper he exhibits the forms assumed by these equations, when the nets are the various isothermal systems of conics.

16. The following theorem is known: Given any integers

$\alpha_1, \cdots, \alpha_p$, $0 \leqq \alpha_\mu < p$, p prime, then there exist polynomials with integral coefficients such that the α form a complete residue system modulo p of the polynomial. If we demand that $f(\mu) \equiv \alpha_\mu \bmod p$ and that $f(x)$ be of degree less than p, the polynomial is uniquely determined modulo p.*

The results obtained by Dr. Kempner may be stated as follows: Let $m = p_1^{\beta_1} \cdots p_\mu^{\beta_\mu}$, and $\gamma_1, \cdots, \gamma_m$ any integers satisfying $0 \leqq \gamma < m$, then there is usually no polynomial $f(x)$ with integral coefficients for which $\gamma_1, \cdots, \gamma_m$ form a complete residue system modulo m. Necessary and sufficient relations between the γ are developed for the existence of a polynomial of which the γ are a complete residue system (without real loss of generality $f(\mu) \equiv \gamma_\mu \bmod m$ may be assumed). When $\gamma_1, \cdots, \gamma_m$ are of such character that they may form a residue system, there is always a polynomial of degree smaller than $F(m)$, where $F(m)$ is defined as the smallest integer such that $[F(m)]!$ contains m as a factor.

It is next shown how the coefficients of the polynomial may be restricted so that to a possible residue system corresponds exactly one polynomial.

For a given modulus m the number of possible residue systems is determined.

The methods of proof are based on the examination of polynomials $\varphi(x)$ with rational coefficients which are congruent to zero modulo a given m for all $x = 0, \pm 1, \pm 2, \cdots$ in inf. and on the introduction of "arithmetical sequences modulo m of a given order," which are obtained from ordinary arithmetical sequences of a given order by reducing all elements and also all differences modulo m. Many related questions are treated.

17. Wilczynski's method of discussion of the projective differential geometry of a geometrical configuration is based upon either a certain set of ordinary differential equations or a certain completely integrable system of partial differential equations. A necessary step in the discussion is the construction of a fundamental set of seminvariants. In the cases in which the basic system of differential equations has one dependent variable this construction has been accomplished by the reduction of the system to a canonical form, the independent coefficients of which form are the fundamental set

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* Zsigmondy, *Monatshefte fur Math. und Physik*, vol. 8 (1897), p. 21.

required. If the basic system of differential equations is a completely integrable system of partial differential equations with one dependent variable, Dr. Nelson shows that this process can be made to yield a pseudo-canonical form, the independent coefficients of which are a simpler fundamental set of seminvariants than the usual process gives.

18. The purpose of the paper by Professor Bliss is to prove the differentiability of the solutions of a system of differential equations with respect to the constants of integration by a method which seems more natural and simpler than those which have hitherto been published. Incidentally a restatement of the so-called imbedding theorem for differential equations is given, a theorem which is frequently applied in the calculus of variations, and which has been useful, and could be made still more so, in many other connections. It is analogous to the fundamental theorem for implicit functions in its statement that a solution of a system of differential equations given in advance is always a member of a continuous family of such solutions. The proofs gain much in simplicity by the use of Peano's matrix notations and his notion of the modulus of a matrix.

19. Professor Ettlinger considers the boundary value problem consisting of the differential equation

$$[K(x, \lambda)u_x]_x - G(x, \lambda)u = 0$$

together with the boundary conditions

$$A_iu(a) - A_i'K(a)u_x(a) - B_iu(b) + B_i'K(b)u_x(b) = 0 \qquad (i = 1, 2),$$

which are not self-adjoint. The types of non-self-adjoint conditions are classified and the existence of at least one characteristic number is established for two important cases.

20. The class of functional equations considered by Professor Carmichael in this note includes linear homogeneous differential, difference and q-difference equations. For the general linear homogeneous differential equation

$$a_0D^ny + a_1D^{n-1}y + \cdots + a_ny = 0, \tag{1}$$

in which D denotes d/dx, a solution y_1 is said to be r-fold if

$y_1, xy_1, \cdots, x^{r-1}y_1$ are all solutions while $x^r y_1$ is not a solution. If r is greater than unity, the solution is said to be repeated. If y_1 is a repeated solution, then it must also satisfy the equation

$$na_0D^{n-1}y + (n-1)a_1D^{n-2}y + \cdots + a_{n-1}y = 0,$$

that is, the equation obtained from (1) by formal differentiation with respect to D. The first elements of the theory of repeated solutions of (1) and a certain more general class of equations thus suggested is developed on a simple postulational basis.

ARNOLD DRESDEN,
Secretary of the Section.

ELEMENTARY INEQUALITIES FOR THE ROOTS OF AN ALGEBRAIC EQUATION.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, October 27, 1917.)

1. LET us write the general algebraic equation in each of the following forms:*

$$\begin{aligned} &x^n = a_1x^{n-1} + a_2^2x^{n-2} + a_3^3x^{n-3} + \cdots + a_n^n, \\ (1)\qquad &x^n = c_{n1}\alpha_1x^{n-1} + c_{n2}\alpha_2^2x^{n-2} + \cdots + c_{nn}\alpha_n^n, \\ &x^n = \beta_1x^{n-1} + \beta_2x^{n-2} + \cdots + \beta_n, \end{aligned}$$

where

$$a_i^i = c_{ni}\alpha_i^i = \beta_i \qquad (i = 1, 2, \cdots, n),$$

and $c_{n1}, c_{n2}, \cdots, c_{nn}$ denote the binomial coefficients for the power n.

If we let X denote the greatest absolute value of a root of equation (1) and let α denote the greatest absolute value of the quantities $|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|$, then, as was shown by Carmichael and Mason,† we have $X \geqq \alpha$, the equality sign holding in a special case, namely, for the equation $(x - \alpha)^n = 0$. Birkhoff, in the same volume,* reproduced Carmichael and Mason's proof of this theorem and obtained further an upper bound for X, so that on combining the two theorems one has

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* The fruitful and convenient notation employed in the first equation was suggested to me by my friend and colleague, Dr. A. J. Kempner.

† This BULLETIN, vol. 21 (1914), pp. 14–22. Carmichael and Mason stated the theorem for the equation whose roots are the reciprocals of those of (1).

$$\alpha \leqq X \leqq \frac{\alpha}{2^{1/n} - 1}. \tag{2}$$

The theorem in (2) was also proved independently by Jensen.†

Notwithstanding the fact that the upper bound in (2) is attained in a special case, namely, that of the equation $2x^n - (x + \alpha)^n = 0$, it is far greater than needful in the case of certain types of equation, as, for instance, the binomial equation $x^n = \alpha^n$. Here the roots are in absolute value equal to α, while the Birkhoff-Jensen upper bound increases indefinitely as n increases. See an extension of the Birkhoff-Jensen result in § 6 below.

The upper bound in (2), as well as several related results, was obtained by Fujiwara‡ by an ingenious method which we shall employ below in the derivation of several theorems.

2. Carmichael and Mason (loc. cit., pages 21–22) also gave an upper bound to the roots of (1), showing indeed that

$$X \leqq \{1 + |\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2\}^{\frac{1}{2}}. \tag{3}$$

Probably this bound is never attained. But if we denote by B the second member of (3) and choose ϵ any positive quantity, however small, then we can find equations all of whose roots are in absolute value between $B(1 - \epsilon)$ and B. In fact it is sufficient to take a binomial equation $x^n = \beta_n$ and choose β_n sufficiently large. Thus the upper bound in (3) is relatively close, in certain cases at least, when the upper bound in (2) is too great by as large a factor as one pleases. On the other hand, the bound in (3) is unsatisfactory in that it is greater than unity however small the coefficients of the equation may be. Fujiwara§ has employed (3) itself to obtain another bound not having this disadvantage, namely, that given in the relation

$$X \leqq \left\{ \frac{n^n}{(n - 1)^{n-1}} (|\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2) \right\}^{\frac{1}{2n}}, \tag{4}$$

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* Ibid., vol. 21 (1915), pp. 494–495.

† *Nyt Tidsskrift*, vol. 26A (1915), pp. 6–13. Some other related results are also given by Jensen.

‡ *Tôhoku Math. Journal*, vol. 10 (1916), pp. 167–171.

§ *Tôhoku Math. Journal*, vol. 8 (1915), pp. 82–85.

this being valid provided that the condition

$$(5)\qquad |\beta_1|^2+|\beta_2|^2+\cdots+|\beta_n|^2<\frac{1}{n-1}$$

is satisfied. He (loc. cit.) has also generalized (3) by showing that, when the roots $x_1, x_2, \cdots, x_n$ of (1) are arranged so that $|x_1| \leqq |x_2| \leqq \cdots \leqq |x_n|$, we have

$$(6)\quad |x_p| \leqq |x_1x_2\cdots x_p|^{1/p} \leqq \{1+|\beta_1|^2+|\beta_2|^2+\cdots+|\beta_n|^2\}^{\frac{1}{2}p}.$$

Kuniyeda* has given interesting generalizations of (3) and (4), showing that

$$(7)\quad X \leqq \{1+(|\beta_1|^{1+p}+|\beta_2|^{1+p}+\cdots+|\beta_n|^{1+p})^{1/p}\}^{p/(1+p)},$$

$$(8)\quad X \leqq \left\{\frac{n^n}{(n-1)^{n-1}}(|\beta_1|^{1+p}+|\beta_2|^{1+p}+\cdots+|\beta_n|^{1+p})^{1/p}\right\}^{p/n(1+p)},$$

where p denotes any positive number, the latter relation being certainly valid only when

$$(9)\qquad (|\beta_1|^{1+p}+\cdots+|\beta_n|^{1+p})^{1/p} \leqq \frac{1}{n-1}.$$

Putting $p = 1$ in (7) and (8), we have the special cases (3) and (4).

The theorem in (3) and certain immediate related results have also been given by S. B. Kelleher† (without reference).

For certain other discussions of inequalities related to those given above reference may be made to papers by Kakeya‡ and Kojima.§ A new proof of some of the results in the latter paper will be given in the sequel.‖

3. The principal object of the present paper is to derive (in §§ 5–10) numerous elementary inequalities for the greatest¶ absolute value X of a root of equation (1). In every case the

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* Ibid., vol. 9 (1916), pp. 167–173; vol. 10 (1916), pp. 187–188.

† *Journ. de Math.* (7), vol. 12 (1916), pp. 168–171.

‡ *Tôhoku Math. Journal*, vol. 2 (1912), p. 140; vol. 3 (1913), p. 23.

§ Ibid., vol. 5 (1914), p. 54; vol. 11 (1917), p. 119.

‖ For two other very interesting inequalities on the roots of algebraic equations see Fejér, *Math. Annalen*, vol. 65 (1908), pp. 413–423 and Lévy, *Nouvelles Annales de Math.* (3), vol. 11 (1892), pp. 147–148.

¶ In an obvious manner these yield corresponding inequalities for the *least* absolute value of a root of equation (1).

method is altogether elementary. The results generalize a number of the known inequalities and contain several new ones of interest. It is clear that numerous other inequalities may be obtained by a further use of the elementary methods here employed.

4. The formula given by Carmichael and Mason (loc. cit., pages 21–22) for the exact value of X may be put into a shape more convenient for certain purposes than the determinant form in which they presented it. Let us consider the equation

$$1 - a_1 z - a_2^2 z^2 - \cdots - a_n^n z_n = 0,$$

whose roots are the reciprocals of the roots of (1), and let ξ denote a root of least absolute value of this equation. Write

$$(10) \qquad \frac{1}{1 - a_1 z - a_2^2 z^2 - \cdots - a_n^n z^n} = 1 + c_1 z + c_2 z^2 + \cdots.$$

Now, the circle of convergence of the power series in the second member of this equation passes through the point ξ. Its radius of convergence is therefore equal to $|\xi| = 1/X$. Therefore by a well-known property of power series we have

$$(11) \qquad X = \limsup_{m=\infty} \sqrt[m]{|c_m|}.$$

But from (10) it is clear that we have

$$1 + (a_1 z + a_2^2 z^2 + \cdots + a_n^n z^n) + (a_1 z + \cdots + a_n^n z^n)^2 + \cdots$$
$$= 1 + c_1 z + c_2 z^2 + \cdots.$$

Expanding the parenthesis quantities in the first member by means of the multinomial formula and equating coefficients of like powers of z, we have

$$(12) \qquad c_m = \Sigma \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1!\, t_2! \cdots t_n!} a_1^{t_1} a_2^{2t_2} a_3^{3t_3} \cdots a_n^{nt_n},$$

where the summation extends over all the non-negative values $t_1, t_2, \cdots, t_n$ for which $t_1 + 2t_2 + 3t_3 + \cdots + nt_n = m$. Taking (11) and (12) together we have the desired explicit formula for X in terms of the coefficients of equation (1).

Though the value of X given in (11) and (12) is exact it is far too unwieldy to be useful in applications.

By comparing the second member of (12) with the multinomial expansion of $(|a_1| + |a_2| + \cdots + |a_n|)^m$ and making use of (11) one may readily show that

$$X \leqq |a_1| + |a_2| + \cdots + |a_n|. \tag{13}$$

(A more elementary proof of this formula is given in § 7 below.) It is obvious that the upper bound in (13) is actually attained in the case of binomial equations.

5. Employing the method of Fujiwara,* we start with the inequality

$$\rho^n \leqq |a_1|\rho^{n-1} + |a_2|^2\rho^{n-2} + \cdots + |a_n|^n, \tag{14}$$

which must obviously be satisfied by the absolute value ρ of any root of equation (1). If we suppose that

$$\rho^n > \lambda_k |a_k|^k \rho^{n-k} \qquad (k = 1, 2, \cdots, n),$$

while the positive quantities λ have the sum of their reciprocals not greater than unity, we shall evidently be led to a contradiction with (14). Hence we have the central theorem of Fujiwara's paper, namely,†

$$X \leqq \max \{\lambda_k^{1/k}|a_k|\} \quad (k = 1, 2, \cdots, n), \tag{15}$$

provided that the positive quantities λ satisfy the relation

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} \leqq 1. \tag{16}$$

Let us consider the question as to what values of the λ's will make the bound afforded by (15) the lowest possible. It is easy to see that they are the values which render equal the n quantities in the second member of (15) and are such that the equality sign in (16) affords a valid relation. Supposing the λ's chosen in this way, let us write

$$\sigma = \lambda_k^{1/k}|a_k| \qquad (k = 1, 2, \cdots, n).$$

Solving these equations for λ_k and substituting in (16) (with the sign of inequality replaced by the equality sign), we have

$$\sigma^n = |a_1|\sigma^{n-1} + |a_2|^2\sigma^{n-2} + \cdots + |a_n|^n.$$

It is easy to see that this equation has a single positive root. Hence we are led to the following theorem, which we state in the notation of the third form of equation (1):

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[*] *Tôhoku Math. Journal*, vol. 10 (1916), pp. 167–171.

[†] The theorem obviously remains true if the equality sign is expunged from each inequality in it.

No root of the equation

$$x^n = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$$

is in absolute value greater than the positive root of the equation

$$\sigma^n = |\beta_1|\sigma^{n-1} + |\beta_2|\sigma^{n-2} + \cdots + |\beta_n|.$$

It is evident that this bound is actually attained in the case of any equation in which the β's are positive or zero. Moreover, it is clear that this is the lowest bound that can be obtained by the direct use of equation (14). Furthermore, it is obvious that a close approximation to this bound is readily obtained in the case of any numerical equation.

6. From (16) it follows that $\lambda_k \geqq 1$; and thence, when (16) is satisfied, we see that

$$X \leqq \max \{\lambda_k^{l_k}|a_k|\} \quad (k = 1, 2, \cdots, n), \tag{17}$$

where l_k is any real quantity not less than $1/k$.

Applying this to an equation of $t + 1$ terms, we have the following result:

The maximum absolute value X of a root of the equation

$$x^n = \gamma_1^{n-n_1}x^{n_1} + \gamma_2^{n-n_2}x^{n_2} + \cdots + \gamma_{t-1}^{n-n_{t-1}}x^{n_{t-1}} + \gamma_t^n, \tag{18}$$

where $n, n_1, \cdots, n_{t-1}$ form a decreasing sequence of positive integers, satisfies the relation

$$X \leqq \max \{\lambda_k^{1/k}|\gamma_k|\} \quad (k = 1, 2, \cdots, t), \tag{19}$$

provided that the positive quantities λ are subject to the condition

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_t} \leqq 1. \tag{20}$$

If we consider the question as to what values of the λ's will make the bound afforded by (19) the lowest possible we shall be led by the method of § 5 to the following theorem (giving such lowest bound):

No root of equation (18) *is in absolute value greater than the positive root of the equation*

$$s^t = |\gamma_1|s^{t-1} + |\gamma_2|^2 s^{t-2} + |\gamma_3|^3 s^{t-3} + \cdots + |\gamma_t|^t.$$

Taking $\lambda_k = t$ in (19), we see that we have for equation

(18) the relation

$$X \leqq \max \{t^{1/k}|\gamma_k|\} \qquad (k = 1, 2, \cdots, t),$$

a result given by Cauchy for the case $t = n$.

Next, let us set

$$\lambda_k = \frac{1}{c_{tk}(\sqrt[t]{2} - 1)^k}.$$

Then (20) is a valid relation. Hence we have through (19) the inequality

$$X \leqq \max \left\{ \frac{|\gamma_k|}{c_{tk}(\sqrt[t]{2} - 1)} \right\} \qquad (k = 1, 2, \cdots, t).$$

The special case $t = n$ is the Birkhoff-Jensen upper bound given in (2). The method of proof is that employed for this special case by Fujiwara in the *Tôhoku Mathematical Journal,* volume 10. For an equation of high degree and relatively few terms the bound is much closer than that of Birkhoff and Jensen.

7. In relation (17) we may choose special values of the quantities l_k so as to obtain the relation

$$X \leqq \text{Max}\, \{\lambda_1|a_1|, \lambda_2|a_2|, \cdots, \lambda_{k_1}|a_{k_1}|, \lambda_{k_1+1}^{1/2}|a_{k_1+1}|, \cdots,$$
$$\lambda_{k_2}^{1/2}|a_{k_2}|, \lambda_{k_2+1}^{1/3}|a_{k_2+1}|, \cdots, \lambda_n^{1/t}|a_n|\},$$

provided that the positive quantities λ satisfy (16). If we consider the question as to what values of the λ's will make the bound afforded by this relation the lowest possible we shall be led by the method of § 5 to the following theorem (giving such lowest bound):

No root of equation (1) *is in absolute value greater than the positive root of the equation*

$$(21) \quad r^t = (|a_1| + \cdots + |a_{k_1}|)r^{t-1} + (|a_{k_1+1}|^2 + \cdots$$
$$+ |a_{k_2}|^2)r^{t-2} + \cdots + (|a_{k_{t-1}+1}|^t + \cdots + |a_n|^t).$$

It is clear that one may prove in a similar way that

No root of equation (1) *is in absolute value greater than the positive root of the equation*

$$r^n = (|a_1| + \cdots + |a_{k_1}|)r^{n-1} + (|a_{k_1+1}|^{k_1+1} + \cdots$$
$$(22) \qquad + |a_{k_2}|^{k_1+1})r^{n-k_1-1} + \cdots + (|a_{k_{t-1}+1}|^{k_{t-1}+1} + \cdots$$
$$+ |a_n|^{k_{t-1}+1})r^{n-k_{t-1}-1}.$$

In the former of the last two results, take $t = 1$; then we have

$$X \leqq |a_1| + |a_2| + \cdots + |a_n|,$$

a relation obtained in § 4 above by less elementary means.

In (21) let $t = 2$ and write

$$|a_1| + \cdots + |a_{k_1}| = s_1, \quad |a_{k_1+1}|^2 + \cdots + |a_n|^2 = s_2.$$

Then we see that

$$X \leqq \tfrac{1}{2}\{s_1 + \sqrt{s_1^2 + 4s_2}\}.$$

In particular, if $a_1 = 0$ we may take $k_1 = 1$ and show that

$$X \leqq \sqrt{|a_2|^2 + |a_3|^2 + \cdots + |a_n|^2}.$$

More generally, if $a_1 = 0 = a_2 = \cdots = a_{\mu-1}$, we have readily from (22) the relation

$$X \leqq \sqrt[\mu]{|a_\mu|^\mu + |a_{\mu+1}|^\mu + \cdots + |a_n|^\mu}.$$

We may also apply to (21) and (22) the theorems already proved for equation (1) and thus obtain other results for equation (1) itself. Thus through use of (15) and (22) we see that

$$X \leqq \max\{\lambda_1(|a_1| + \cdots + |a_{k_1}|), [\lambda_2(|a_{k_1+1}|^{k_1+1} + \cdots \\ + |a_{k_2}|^{k_1+1})]^{1/(k_1+1)}, \cdots\}$$

provided that the positive quantities λ satisfy relation (20).

8. In (19) let us take

$$\lambda_k = \frac{1}{|\gamma_k|^k}(|\gamma_1| + |\gamma_2|^2 + \cdots + |\gamma_t|^t).$$

Then (20) is satisfied and from (19) we see that for equation (18) we have

$$X \leqq \max\{|\gamma_1| + |\gamma_2|^2 + \cdots + |\gamma_t|^t\}^{1/k} \quad (k = 1, 2, \cdots, t).$$

Hence for equation (18) we have

$$(23) \qquad \begin{aligned} X &\leqq \left\{\sum_{i=1}^{t} |\gamma_i|^i\right\}^{1/t} \text{ if } \sum_{i=1}^{t} |\gamma_i|^i \leqq 1; \\ X &\leqq \sum_{i=1}^{t} |\gamma_i|^i \qquad \text{ if } \sum_{i=1}^{t} |\gamma_i|^i \geqq 1. \end{aligned}$$

The method of proof here employed is identical with that used by Fujiwara (loc. cit.) for the special case $t = n$. From the first relation in (23) we see readily that

$$X \leqq \left\{ t \sum_{i=1}^{t} |\gamma_i|^{2i} \right\}^{1/2t} \quad \text{if} \quad \sum_{i=1}^{t} |\gamma_i|^{i} \leqq 1,$$

since

$$\left(\sum_{i=1}^{t} |\gamma_i|^{i} \right)^2 \leqq t \sum_{i=1}^{t} |\gamma_i|^{2i}$$

as one may readily prove.

9. The theorem of Fujiwara stated in relation (15) will also yield the upper bound given in (3). For the third form of equation (1) relation (15) may be written

$$X \leqq \max \{\lambda_k |\beta_k|\}^{1/k} \quad (k = 1, 2, \cdots, n),$$

the positive quantities λ being again subject to relation (16). For the proof of (3) it is obviously sufficient to show that (16) is satisfied if the λ's have the values

$$\lambda_k = \frac{1}{|\beta_k| \nu^k}, \quad \text{where} \quad \frac{1}{\nu} = \{1 + |\beta_1|^2 + \cdots + |\beta_n|^2\}^{1/2}.$$

For the moment denote the first member of (16) by S. Then we have

$$\begin{aligned} S &= |\beta_1|\nu + |\beta_2|\nu^2 + \cdots + |\beta_n|\nu^n \\ &\leqq \{|\beta_1|^2 + \cdots + |\beta_n|^2\}^{1/2}\{\nu^2 + \nu^4 + \cdots + \nu^{2n}\}^{1/2} \\ &\leqq \left\{ \frac{1}{\nu^2} - 1 \right\}^{1/2} \{\nu^2 + \nu^4 + \cdots + \nu^{2n}\}^{1/2} \\ &\leqq \{1 - \nu^{2n}\}^{1/2} < 1, \text{ since } \nu < 1. \end{aligned}$$

Hence (16) is satisfied and (3) is proved.

Since (3) is proved by aid of (15) it is clear that it can never afford a lower bound for X than that given in the italicized theorem of § 5.

In a similar way one may prove the generalization (7) due to Kuniyeda. The method is essentially that of Kuniyeda's memoir. It is easy to see further that (7) can never afford a lower bound for X than that given in the italicized theorem in § 5.

10. From the third form of equation (1) we have

$$|x|^{n-k}|x^k - \beta_1 x^{k-1} - \cdots - \beta_k| \leqq |\beta_{k+1}|\cdot|x|^{n-k+1} + \cdots + |\beta_n|.$$

Hence if

$$|x^k - \beta_1 x^{k-1} - \cdots - \beta_k| \geqq r \quad (r > 0),$$

we have

$$|x|^{n-k} \leqq \frac{1}{r}\,|\beta_{k+1}|\cdot|x|^{n-k-1} + \cdots + \frac{1}{r}\,|\beta_n|.$$

Therefore the last inequality will certainly be satisfied if

$$|x|^k \geqq |\beta_1|\cdot|x|^{k-1} + \cdots + |\beta_{k-1}|\cdot|x| + (|\beta_k| + r).$$

Thence it is easy to see that

The greatest absolute value X *of a root of equation* (1) *satisfies the relation*

$$X \leqq \max\{\rho_1, \rho_2\},$$

where ρ_1 *and* ρ_2 *are the positive roots of the equations*

$$\rho^k = |\beta_1|\rho^{k-1} + \cdots + |\beta_{k-1}|\rho + (|\beta_k| + r), \tag{24}$$

$$\rho^{n-k} = \frac{1}{r}\,|\beta_{k+1}|\rho^{n-k-1} + \cdots + \frac{1}{r}\,|\beta_n|, \tag{25}$$

respectively, r *being any positive quantity and* k *any integer less than* n.

This theorem itself may obviously be applied to either or both of equations (24) and (25); so that an upper bound to X may be obtained in terms of the solutions of equations of as low degree as one wishes. Thus, as a special case, we may take the equations to be linear and thus obtain the following result (found by Kojima* in a less elementary way):

$$X \leqq \max\left\{|\beta_1| + r_1,\ \frac{|\beta_k|}{r_1 r_2 \cdots r_{k-1}} + r_k \quad (k = 2, \cdots, n-1),\ \frac{|\beta_n|}{r_1 r_2 \cdots r_{n-1}}\right\},$$

where $r_1, r_2, \cdots, r_{n-1}$ are any positive quantities.

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* *Tôhoku Math. Journal*, vol. 11 (1917), pp. 119–127. The first two special cases noted had been given earlier by the same author, ibid., vol. 5 (1914), pp. 54–60.

Taking each r equal to unity, we have the interesting special case

$$X \leqq \max\{1 + |\beta_1|, 1 + |\beta_2|, \cdots, 1 + |\beta_{n-1}|, |\beta_n|\}.$$

Again, it is easy to show also that

$$X \leqq \max\{|\alpha| + |\beta|, |\beta| + |\gamma|, \cdots, |\lambda| + |\mu|, |\mu|\},$$

where the quantities $\alpha, \beta, \cdots, \mu$ are defined by the relations

$$\alpha = \beta_1, \beta^2 = \beta_2, \beta\gamma^2 = \beta_3, \beta\gamma\delta^2 = \beta_4, \cdots, \beta\gamma\cdots\lambda\mu^2 = \beta_n.$$

Through other special choices of the quantities r numerous rather elegant special inequalities may be obtained, several of which are given explicitly by Kojima (loc. cit.).

In case some of the coefficients of equation (1) are zero it may be preferable to employ the second italicized theorem in § 6 and apply to it the principal theorem of this section (or the special cases of the latter).

It is clear that other general formulas may be found for upper bounds to X by employing a sequence of equations each of which is linear or quadratic, such sequence arising by the repeated application of the theorem of this section.

UNIVERSITY OF ILLINOIS,
October, 1917.

THE SOLUTION OF THE WAVE EQUATION BY MEANS OF DEFINITE INTEGRALS.

BY PROFESSOR H. BATEMAN.

THE wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \tag{1}$$

is the oldest member of the family of partial differential equations, and although he was born without the second and third terms he soon acquired these and played a prominent part in mathematical physics at a time when very few partial differential equations had become famous. With the advent of the mathematical theory of elasticity and Maxwell's electromagnetic theory of light he gained a new lease of life and more than doubled his vitality, so that his power is strongly felt even at the present time. In a spirit of reverence for age and valuable service we can still learn much by consulting the old pioneer; not only is he able to throw light on the much discussed subject of the nature of electricity and the general theory of partial differential equations, but he is leaving us a legacy of unsolved problems concerning the diffraction of sound and electric waves and these will tax the ingenuity of mathematicians for years to come. It is often said that Maxwell's equations do not tell us anything about the structure of the æther, but this view may not be correct. Although the equations themselves do not give any clear indications of the properties of the æther, yet the solutions of the equations may give some information. It must be realized that when a physical problem is reduced to differential equations some of the features of the problem are eliminated in the process and very often these features are of great physical interest. Unfortunately we cannot retrace the process of elimination and the only process of reconstruction open to us is to make an exhaustive study of the solutions of the differential equations and to pay particular attention to cases in which a very general solution is built up from elementary solutions of a particularly simple type.

In the solution of diffraction problems the modern tendency seems to be to make use of integrals taken along complex paths in representing the required solutions of the wave equation.* This method is very powerful and beautiful, but to obtain a visualization of the physical processes which actually take place it seems desirable to make use of integrals taken along real paths and some progress has already been made in this direction. Again in A. W. Conway's derivation† of the potentials for a moving point charge of electricity use is made of contour integrals of type

$$(2) \quad \int \frac{f(\tau)d\tau}{[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2-c^2(t-\tau)^2},$$

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* See, for instance, A. Sommerfeld, *Math. Ann.*, Bd. 45 (1894), Bd. 47 (1896); *Zeitschr. für Math. u. Phys.*, Bd. 46 (1901); H. M. Macdonald, Electric Waves (1902); *Proc. London Math. Soc.*, ser. 2, vol. 14 (1915), p. 410; H. S. Carslaw, *Proc. London Math. Soc.*, ser. 1, vol. 30 (1898), p. 121; *Math. Ann.*, Bd. 75 (1914), p. 133; T. J. I'A. Bromwich, *Proc. London Math. Soc.*, ser. 2, vol. 14 (1915), p. 450; F. J. W. Whipple, ibid., vol. 16 (1917), p. 94.

† *Proc. London Math. Soc.*, ser. 2, vol. 1 (1903).

but it has been shown recently* that a similar derivation is possible by taking the integral (2) between certain consecutive real limits τ_1 and τ_2, each of which is defined by a pair of equations of type

$$[x - \alpha(\theta)]^2 + [y - \beta(\theta)]^2 + [z - \gamma(\theta)]^2 = c^2(t - \theta)^2, \quad t \geqq \theta,$$

$$[\alpha(\theta) - \xi(\tau)]^2 + [\beta(\theta) - \eta(\tau)]^2 + [\gamma(\theta) - \zeta(\tau)]^2 = c^2(\theta - \tau)^2, \quad \theta \geqq \tau.$$

It is easy to verify that the definite integral is in this case a wave function;† it is also easy to verify that an integral of the above type represents a wave function when the limits are functions of type τ which is defined by an equation such as

$$l(\tau)[x - \xi(\tau)] + m(\tau)[y - \eta(\tau)] + n(\tau)[z - \zeta(\tau)] = c(t - \tau),$$

where

$$[l(\tau)]^2 + [m(\tau)]^2 + [n(\tau)]^2 \equiv 1.$$

This brings us to the consideration of the following problem: If $f(x, y, z, t, \theta)$ is a wave function for all values of the parameter θ, the definite integral

$$V = \int_{\theta_1}^{\theta_2} f(x, y, z, t, \theta) d\theta$$

is certainly a wave function when θ_1 and θ_2 are suitable constants, but it may also be a wave function when θ_1 and θ_2 are certain functions of x, y, z and t. The problem is to find a suitable pair of functions θ_1 and θ_2 when $f(x, y, z, t, \theta)$ is given.

Let us commence by considering the case when f is a wave function of type‡ $\gamma F(\alpha, \beta)$ where F is an arbitrary function of its two arguments and α, β, γ are functions of x, y, t and z. This type of wave function may be regarded as more fundamental even than the familiar function

$$[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 - c^2(t - \tau)^2]^{-1},$$

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* H. Bateman, *Tohôku Math. Journal*, vol. 13.

† The analogous solution of the wave equation in two dimensions may be used to solve the problem of the diffraction by a half plane of the waves from a moving line source of sound, the line being parallel to the edge of the diffracting plane.

‡ The problem of finding all possible forms for α, β and γ is solved in this BULLETIN, vol. 21 (1915), p. 299, and in *Amer. Journal of Mathematics*, April, 1915. We may write

$$z - ct = \phi(\alpha, \beta) + (x + iy)\chi(\alpha, \beta),$$

$$z + ct = \psi(\alpha, \beta) - (x - iy)/\chi(\alpha, \beta).$$

which gives rise to the integrand in (2). In electromagnetic theory, for instance, if E and H denote electric and magnetic intensities respectively, the æthereal field in which the complex vector $M \equiv H + iE$ is represented by the expressions

$$M = F(\alpha, \beta)(\nabla\alpha \times \nabla\beta) \equiv \frac{i}{c} F(\alpha, \beta) \left[\frac{\partial\beta}{\partial t} \nabla\alpha - \frac{\partial\alpha}{\partial t} \nabla\beta \right]$$

is a *pure radiant field*, while the quantities α and β are constant along the rays which are the rectilinear paths of the energy. It has been shown that the field of a point charge of electricity can be regarded as the limiting case of two superposed radiant fields of the above description in each of which electricity flows with the velocity of light along certain pairs of singular rays. The singular rays in the two fields overlap and there is a cancelling of electricity except on two segments which shrink to a point as the limit is approached. This suggests that the æther may consist of particles of electricity and electric doublets travelling along straight lines with the velocity of light. In this connection it may be worth while to point out that the rays are bicharacteristics* of the partial differential equations while α and β are functions which satisfy the partial differential equation of the characteristics of equation (1).

Let us now make α and β depend on a parameter θ as well as on x, y, t and z; forming the definite integral

$$M = \int_{\theta_1}^{\theta_2} F(\alpha, \beta, \theta)(\nabla\alpha \times \nabla\beta) d\theta$$

$$\equiv \frac{i}{c} \int_{\theta_1}^{\theta_2} F(\alpha, \beta, \theta) \left(\frac{\partial\beta}{\partial t} \nabla\alpha - \frac{\partial\alpha}{\partial t} \nabla\beta \right) d\theta$$

and writing

$$\frac{\partial\alpha}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial\alpha}{\partial y}\frac{\partial\theta}{\partial y} + \frac{\partial\alpha}{\partial z}\frac{\partial\theta}{\partial z} - \frac{1}{c^2}\frac{\partial\alpha}{\partial t}\frac{\partial\theta}{\partial t} \equiv (\alpha, \theta),$$

we find that

$$\text{rot } M + \frac{i}{c}\frac{\partial M}{\partial t} \equiv \{(\beta, \theta)\nabla\alpha - (\alpha, \theta)\nabla\beta\} F(\alpha, \beta, \theta) \Big|_{\theta=\theta_1}^{\theta=\theta_2}.$$

The right-hand side is clearly zero if θ_1 and θ_2 are solutions of the two equations

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* The theory of bicharacteristics is expounded in Hadamard's Propagation des Ondes.

$$(\alpha, \theta) = 0 \quad \text{and} \quad (\beta, \theta) = 0.$$

Now $(\alpha, \alpha) = 0$, $(\alpha, \beta) = 0$ and $(\beta, \beta) = 0$, hence the above equations may be satisfied by taking θ_1 and θ_2 to be roots of the equations

$$G_1(\alpha, \beta, \theta) = 0 \quad \text{and} \quad G_2(\alpha, \beta, \theta) = 0$$

respectively.* It should be noticed in passing that θ_1 and θ_2 are solutions of the characteristic equation $(\theta, \theta) = 0$. Since

$$\operatorname{rot} M + \frac{i}{c}\frac{\partial M}{\partial t} = 0$$

it follows that each component of M satisfies the wave equation and so we have an important class of wave functions represented by definite integrals in which the limits are functions of x, y, z and t. It should be noticed that if we substitute a definite integral of type

$$V = \int_{\theta_1}^{\theta_2} \gamma F(\alpha, \beta, \theta) d\theta$$

in the wave equation we obtain a system of partial differential equations for θ_1 and θ_2 and these are difficult to solve directly. The above method, however, indicates the nature of the complete solution of these equations. It also emphasizes the advantage which is gained by using a system of partial differential equations of the first order instead of the second order equation (1). This plan might be adopted with advantage in other cases; it may be recalled also that the first order equations may be used more profitably than (1) in the determination of all the functions of type α and β.

The theorem that has just been proved is of wide application. We know for instance that a wave function of type $\gamma F(\alpha, \beta)$ may be obtained by writing

$$\gamma = \frac{1}{R}, \quad \alpha = t - \frac{R}{c}, \quad \beta = \frac{x - iy}{R + z},$$

where R is the distance from the origin. We may generalize this solution by shifting the origin to the point $\xi(\theta)$, $\eta(\theta)$,

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* G_1 and G_2 are arbitrary functions of α, β and θ.

$\zeta(\theta)$ and we then have the result that if

$$r^2 = [x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2,$$

$$\alpha = t - \frac{r}{c}, \quad \beta = \frac{x - \xi(\theta) - i[y - \eta(\theta)]}{r + [z - \zeta(\theta)]},$$

the definite integral

$$V = \int_{\theta_1}^{\theta_2} F(\alpha, \beta, \theta) \frac{d\theta}{r}$$

satisfies the wave equation, provided θ_1 and θ_2 are roots of the equations

$$G_1(\alpha, \beta, \theta) = 0 \quad \text{and} \quad G_2(\alpha, \beta, \theta) = 0$$

respectively. The case in which F, G_1 and G_2 are independent of β is of special interest. V may then be regarded as the velocity potential of a chain of sources of sound each of which is only active for a certain interval of time which may be different for different sources.

THROOP COLLEGE OF TECHNOLOGY,
PASADENA,
October 6, 1917.

LITERATURE OF PURE MATHEMATICS.

Historical Introduction to Mathematical Literature. By G. A. MILLER. New York, Macmillan, 1916. 14 + 302 pp. Price $1.60.

IT is difficult to overestimate the extent of inspiration which may emanate from interesting exposition of problems and wonders of science in a form intelligible to those who are not deeply versed in the subject. How much richer must be the intellectual outlook of thousands throughout the world who have read: Science and Hypothesis, The Value of Science, and Science and Method! Is it hard to believe that the future historian may some day tell us that the very notable stage of advancement of astronomy in America in 1910 was not a little due to Simon Newcomb's remarkable gifts of popularization of his subject, exercised through written and spoken word in magazine, book, society, and congress during a long series of years—educating and inspiring the people to a frame of mind which called for organization, equipment, and cooperation greater than had been?

For many years Professor Miller has been one of a small group of Americans who have very frequently written and spoken with authority in the interests of the mathematical amateur and inquiring general student. There can be little doubt of the success of their efforts to arouse in America more widespread enthusiasm and deeper appreciation of the values involved in the study of mathematics, through discussion of some of its historical, philosophical, geometrical, and analytical aspects. In this connection apart from expository articles (more particularly in the study of groups) Professor Miller has published many articles and addresses attractive not so much on account of their stimulating style, for in this they are sometimes lacking, but rather on account of their timely and highly informational content, very clearly, suggestively, and thoughtfully expressed. We welcome then this unique work,* with similar characteristics, in which much that had been published is coordinated, developed, and expanded, and combined with new features.

We are told that the volume is based on a course of lectures which were designed to supplement the regular work of mathematical courses, and that in it the author (1) "aimed to meet the needs of a textbook for synoptic and inspirational courses which can be followed successfully by those who have not had extensive mathematical training"; and (2) provided a work which "may also be used as a textbook for a first course in the history of mathematics, especially by those teachers who believe with its author that such a first course should largely concern itself with recent mathematical events and developments." The author states that in his presentation he attempted to guide the mathematical student to "points from which he can overlook domains of considerable extent in order that he may be able to form a somewhat independent judgment as regards the regions which he might like to examine more closely." Let us now consider the work in some detail.

The "General Observations" (35 pages) of the first chapter are made under the headings: Changes during the nineteenth century; history of mathematics; developed parts of the history of mathematics; periods in the history of mathematics; usefulness of the history of mathematics; first decade and a half of the twentieth century; American mathematics. Since details of purely domestic interest are recorded in connection with the references to the American Mathematical Society, it is a wonder that one fact of international interest is not mentioned, namely, that since 1913 the society has enjoyed "the possibly unique distinction of holding almost simultaneous meetings in different cities."*

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* G. Loria's Guida allo Studio della Storia delle Matematiche (Milano, 1916), which was published almost simultaneously with this work has very different ideals. See the review by D. E. Smith in this BULLETIN, 1916, vol. 23, pp. 136–139.

The second chapter on "Mathematical Literature in General" is somewhat longer. In the first section on "Types of recent mathematical literature" the discussion is mainly confined to English, French and German periodicals. Except for slips in the dates of the *Cambridge Mathematical Journal* and of the *Cambridge and Dublin Mathematical Journal*, the statements concerning English periodicals are accurate; but surely wrong inferences may be drawn from the two pages devoted to *The Ladies' Diary* (1704–1840) and the *Gentleman's Diary or the Mathematical Repository* (1741–1840).† We are told that the latter "may be regarded as a scientific mathematical journal and it contains a number of important problems and solutions relating to elementary mathematics." And then: "These journals [the diaries] did much to keep alive in England a general interest in mathematics, and they contain a number of theorems relating to elementary mathematics which were rediscovered in later times." Since, for the period they cover, no other English mathematical journal is mentioned, one natural inference from the above would be that there was no other of equal importance. And yet, for example in the eighteenth century‡ there were two journals devoted *wholly*§ to mathematics and certainly of greater scientific significance. The one, *Miscellanea Curiosa Mathematica*,‖ contained William Chapple's notable paper (1746) with its results concerning the distance between the centers of the inscribed and circumscribed circles of a triangle, and other observations anticipating formulas of Euler* and discussions by Poncelet, Steiner, and Jacobi. The other journal, *The Mathematical Repository,*† contains the first enunciation (March, 1799) of the well-known theorem concerning Wallace's line.‡

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* In this BULLETIN, January, 1914, vol. 20, p. 170.

† No reference is made to the amalgamation of the almanacs into *The Lady's and Gentleman's Diary* (1841–1871).

‡ Similar illustrations could be given concerning the nineteenty century.

§ In the latter part of the century not more than a third of the *Diaries* was of a mathematical nature.

‖ Fourteen numbers published in London, 1745–1754.

On page 44 reference might be made to De Morgan's sketch of the "Mathematical Society of Spitalfields" in his Budget of Paradoxes.

Prince Boncompagni's *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche* is listed in a section on "special historical periodicals," and it is stated that a "peculiarity of this periodical is that very extended references are given even when they relate to matters where the literature is easily accessible, and it is probable that these extensive references have tended to decrease the use of the journal, since they are often so numerous as to burden the reader." A much less trivial "peculiarity," and a source of real exasperation to the historical inquirer, is due to the fact that there are probably few (perhaps not even two) sets of the *Bullettino* which are exactly alike throughout. As the periodical was set up and printed in his palace Boncompagni frequently stopped the press after some numbers were printed in order to introduce additions and changes. In this way it is said that several different editions of the same number may be found.§

In the section on "Encyclopedias and other works of reference" a misleading statement has been made in connection with "the Encyclopédie Méthodique Mathématique, in three volumes, by d'Alembert and others, 1784–89." My edition is in at least four volumes, the last of which, Recueil de Planches du Dictionnaire Encyclopedique des Mathématiques, was published in M.DCC.XCVII. I have written "at least" because there are three more volumes published with the following titles: (1) Dictionnaire des Jeux, faisant suite au

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* *Novi Commentarii* . . . *Academiae* . . . *Petropolitanae*, tome 11 (1765) 1767, pp. 103–123.

† London, 1795–1835.

‡ Proposition I, in a paper by William Wallace entitled "Mathematical Lucubrations," vol. II, p. 111. It is implied in a parabola theorem which he published in the fourth number of the *Repository* (October, 1797).

§ Cf. *Isis*, June, 1914, tome II, pp. 133–134. This reference is to an article by G. Sarton containing an annotated list of "Soixante-deux revues et collections relatives à l'histoire des sciences," to which attention might naturally be drawn in such a section as that which we are considering.

Tome III des Mathématiques, 1792; (2) Dictionnaire des Jeux familiers ou des Amusemens de Société, faisant suite au Dictionnaire des Jeux, annexé au tome III des Mathématiques, An V [1797]; (3) Dictionnaire des Jeux mathématiques contenant l'Analyse, les Recherches, les Calculs, les Probabilités et les Tables numériques, publiés par plusieurs célèbres Mathématiciens, relativement aux Jeux de Hasard et de Combinaisons, et suite du Dictionnaire des Jeux, an VII [1799].

In this same section we find also: "The most useful work of reference for the purpose of finding the periodical literature of the nineteenth century is the Subject Index of the Royal Society of London Catalogue of Scientific Papers, volume 1, 1908. . . . Since the beginning of the twentieth century this work is supplemented annually by volume A of the International Catalogue of Scientific Literature." And why not give reference to volume 2, 1909 (Mechanics), to volume 3, 1912–14 (Physics, but with much of value for the mathematician), to volumes B (Mechanics), E (Astronomy), J (Geography, which includes geodesy) and C (Physics)? The fact of the matter is that our author disregards applied mathematics practically altogether, so that a more accurate title of his book would be "Historical Introduction to the Literature of Pure Mathematics."

In another section of the second chapter, on "Mathematical tables," a reference to the work edited by Horsburgh* would have usefully supplemented that to the 1873 Report of the British Association for the Advancement of Science. The following statement is perhaps a trifle misleading: "In 1911 H. Andoyer of the University of Paris published a table giving the logarithms of the trigonometric functions . . . under the title Nouvelles Tables Trigonométriques Fondamentales, as it is expected to become fundamental for other tables." The volume in question was the first of three volumes and it consists mainly of a set of logarithmic tables for sines, cosines, tangents and cotangents. The second and third† volumes (1915–16) are principally occupied with tables of natural trigonometric functions. It seems strange to find no mention in this section of the outstanding works by Crelle,* Peters,† and Bauschinger.‡

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* Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition (1914).

† This volume may have appeared after Professor Miller's book was out.

Other topics of the chapter are: Mathematical societies, international mathematical congresses, periodicals on periodicals and books on books, and collected works.

The third chapter, entitled "General Historical Questions Relevant to Mathematics," includes discussion of such subjects as: Definitions of the term mathematics, divisions and subdivisions of mathematics, some dominant concepts, notation and terminology, errors in mathematical literature, and living mathematicians. As the 1908 and 1916 editions of the Index du Répertoire Bibliographique des Sciences Mathématiques have been published by Delsman in Amsterdam, a change is desirable in a statement on page 89. Another instance, among several, of national rather than international outlook occurs on page 123 where, by indirection, H. H. Stephenson's Who's Who in Science (International) which passed through several editions (1912–1914) is condemned as a source of information as to "mathematical standing," while American Men of Science is characterized as a "much more useful type of work."

In the next three chapters fundamental developments in arithmetic, geometry and algebra are discussed. Arithmetical topics are: Existence of an infinite number of prime numbers with Euclid's proof, sieve of Eratosthenes, irrational quantities and numbers, fundamental operations of arithmetic, systems of notation, and Fermat's last theorem. Is it not misleading to give, without comment, 1228 as the date of the "noted Liber Abaci by Leonardo of Pisa"?

The headings for the chapter in geometry are: The Pythagorean theorem, area of the circle, area and volume of the sphere, regular geometric forms, the triangle. It is recalled that the French and the Germans frequently refer to the

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* A. Crelle, Calculating tables giving the products of every two numbers from one to one thousand and their application to the multiplication and division of all numbers above one thousand. New edition by O. Seeliger, 1908.

† J. Peters: 1. New calculating tables for multiplication and division, by all numbers of from one to four places. 1909. 2. Siebenstellige Logarithmentafel der trigonometrischen Funktionen, für jede Bogensekunde des Quadranten. 1911.

‡ J. Bauschinger and J. Peters, Logarithmic-trigonometrical tables with eight decimal places containing the logarithms of all numbers from 1 to 200,000 and the logarithms of the trigonometric functions for every sexagesimal second of the quadrant. 2 vols. 1910–11.

Pythagorean proposition as "the asses' bridge." Other names might have been mentioned: "Theorem of the married woman" by the Greeks; "Chaise of the little married woman" by the Hindus (Bhascara); and "figure of the bride" by the Persians.* The first edition of J. J. I. Hoffman's collection of proofs of the proposition is referred to; a "zweyte verbesserte und mit einigen neuen Beweisen vermehrte Ausgabe" was published in 1821.

On pages 175–6 it is stated that Gauss proved (Disquisitiones Arithmeticæ, section 7) that "a necessary and sufficient condition that a regular polygon of a prime number of sides can be inscribed in a circle by means of ruler and compass is that this prime be of the form $2^m + 1$." If the reference to the Disquisitiones had been omitted no criticism could have been made of this statement; but as it is, incorrect inferences are likely to be drawn.†

A reference is given to Casey's Sequel to Eculid, 1888, for information concerning the geometry of the triangle. It may be noted that it is this particular edition alone which contains the supplementary chapters on this subject.

Some of the topics in the chapter on Developments in Algebra are: Fundamental theorem of algebra, determinants, numerical equations, domains of rationality, invariants and binomial theorem.

The seventh and last chapter contains brief sketches of twenty-five prominent deceased mathematicians: Euclid, Archimedes, Apollonius of Perga, Diophantus of Alexandria, Viète, Descartes, Fermat, Newton, Leibniz, Euler, Lagrange, Gauss, Cauchy, Steiner, Abel, Hamilton, Galois, Sylvester, Weierstrass, Cayley, Kronecker, Hermite, Cremona, Lie, and Poincaré. In the sketch of Euler it is stated that "he arrived, in 1748, at the celebrated formula

$$e^{ix} = \cos x + i \sin x."$$

i for $\sqrt{-1}$ appears to have been first used by Euler in 1777. In 1722 Roger Cotes gave the equivalent, in words, of $\sqrt{-1}\varphi = \log(\cos\varphi + i\sin\varphi)$. In the sketch of Gauss reference is made to his proof of the fundamental theorem of algebra, to various parts of the Disquisitiones Arithmeticæ, to his work on the theory of orbits and theory of errors, and to his geometric theory of complex numbers. His fundamental monograph in the theory of surfaces is not mentioned, even though the best edition of it was published by Princeton University. It is not accurate to refer to even the major portion of Euclid's book On Divisions (of figures) as "still extant" (page 219). And finally, in the sketches of Hermite and Poincaré our author does not seem to be sufficiently clear as to the use of some French terms: (1) Hermite "became professor also at the École Polytechnique and at the Faculté des Sciences" (page 261); for "Faculté des Sciences" substitute "Université de Paris"; (2) Poincaré "was elected a member of l'Académie des Sciences" (page 269), and "was received as member of l'Académie Française"* (page 271), and "received two prizes from the Paris Academy" (page 272); after each of the first two expressions quoted add "de l'Institut de France," although there is no ambiguity in the second case; for "Paris Academy" (which may be interpreted as having an entirely different meaning from the one intended†) substitute "Institut de France" or "Académie des Sciences de l'Institut de France." It is surely of interest to bring out more clearly the fact that Poincaré was a member of these two Académies of the five composing the Institut de France—the greatest honor which France has in her gift for the élite of her scientists. I am under the impression that J. L. F. Bertrand was the only other scientist who has been similarly honored.

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* Cf. E. Lucas, Récréations mathématiques, tome II, 1883, or 1896, p. 130; E. Fourrey, Curiosités géométriques, Paris, 1907, p. 64.

† In more than one place in Klein's Famous Problems of Geometry misstatements occur in this connection.

An appendix of 15 pages contains an annotated list of bibliographies and encyclopedias, histories, and books on teaching and philosophy of mathematics. The reviewer sees no advantage in giving partial indications of the first, instead of the last, editions of the works mentioned; and even in this respect there is a slip in connection with Cantor's Vorlesungen. The current edition of the Encyklopädie der Elementar-Mathematik by Weber and Wellstein contains four, not three, volumes (page 280).‡

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* This Académie has a "président" but not a "directeur" (p. 271).

† *Cf.* this BULLETIN, March, 1910, vol. 16, p. 329.

‡ There are fairly obvious typographical errors on page 87, line 13; page 130, line 2 from bottom; page 289, line 17; and page 293, line 5. The spelling of the name Čebysév on page 133, line 5, does not agree with that later on the page or in the index; similarly Lobačevský on pages

The Author Index and Subject Index occupy about ten pages.

To sum up: Professor Miller has written a valuable, original, and very entertaining book, containing much out-of-the-way information difficult of access elsewhere. It deals only with pure mathematics, and in this field emphasizes considerably the subjects of groups, theory of numbers, and theory of equations. While the treatment is often, perhaps necessarily, scrappy (by reason of its "synoptic" nature) it has been shown above that the separate sections do not always treat the subject with the breadth to be expected in a volume designed to guide a student to "points from which he can overlook domains of considerable extent." Either the amateur or the professional mathematician must find much of interest within its covers, however, and the work is heartily recommended. R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

SHORTER NOTICES.

A History of Elementary Mathematics with Hints on Methods of Teaching. By FLORIAN CAJORI. New York, The Macmillan Company, 1917. viii + 324 pp. Price $1.75.

Two of the most unsatisfactory literary labors that members of our guild are called upon to undertake are first, the publication of a new edition of an old work, using plates of many years' standing; and second, the review of an edition prepared under such circumstances. Probably the most severe critic of the work in hand is Professor Cajori himself, and if he had been free to do so he would undoubtedly have revised the work more radically than most of his readers would or could undertake to do. The book was written more than twenty years ago, and the world knows more about the subject than it did at that time; it has found better ways of overcoming certain difficulties in the matter of presentation of material; and it has the problem of a history of mathematics better in hand.

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125 and 272 (add to index) differs from the index form; for Graup, page 161, line 4 from bottom, read Graap; for McMahon, page 293, read MacMahon.

Here and above the slips or errors are practically all of somewhat minor importance. As a whole Professor Miller's work is exceedingly accurate.

Professor Cajori's later contributions to the subject, such as those on Oughtred, the slide rule, and the calculus, are of an entirely different type from those embodied in the book under review. This is due not merely to the fact that they are intended for more mature readers; it is due quite as much to the fact that they are the product of a hand that has acquired new cunning and of a mind that has developed new powers.

It is therefore manifestly unfair to review this book as if it had just been written, and the proper course would seem to be to call attention to the changes actually made in the plates and to a few of the problems which have to be met by all who engage in historical writing and which are suggested by this work. The book has been a guide to many who, in the last two decades, have begun their study of the history of mathematics, and such a record is in itself a happy criticism.

Most of the misprints in the first edition have been corrected, although the elimination of such errors is not complete. Less attention has been paid to the correction of dates, a difficult problem at best. The impression that Ahmes wrote in hieroglyphics (page 23) still remains, as does the one that the subtractive principle of notation originated with the Romans (page 8), and the one that Hypsicles and Ptolemy were contemporaries (page 28), but perhaps slips of this kind should be considered charitably in an introductory work.

The additions of chief importance relate to the Maya zero (page 18), the history of logarithms (pages 164–166), the dollar sign (page 223), and certain modern movements in the teaching of elementary mathematics (pages 287–309). Of these, the second is particularly welcome as answering, at least negatively, the question of the originality of Speidell in the matter of the natural logarithms. While the invention of such logarithms is not definitely placed, a step has been made by Professor Cajori towards doing so. The fourth of the leading additions is also welcome, for it sets forth various features of modern geometry that have developed since the first edition of the work appeared. Certain recent developments in teaching, such as the Perry movement and the work of the International Commission and of various American associations, are also mentioned briefly but with sufficient clearness for the information of teachers.

The work is, therefore, something besides a mere reprint, and will serve a good purpose not only for those who are beginning their reading in the history of mathematics but also for those who are interested in the teaching of the subject.

As to the general questions suggested by the work, three may profitably be considered: First, how shall Oriental names be transliterated? If we take the Suter list as given in the Abhandlungen zur Geschichte der Mathematik, which is certainly the most scientific one that has been worked out in mathematics, we run into such a difficulty as that of the sound denoted by the English J. In this respect the scheme is not convenient for English, French, and Spanish readers. If this is settled, what shall be the method adopted for Chinese names? Shall Chu be Chou, Tschu, Tscheou, or Chiu, and shall we have the Yih King, the Yi Ching, or the I Ching as the title of a well-known classic? The same difficulty arises with respect to the languages of India, and indeed of all countries not using the Latin letters. In the case of countries where these letters are used, we can easily fix a satisfactory rule, namely, to spell the name as was most commonly done by the individual, as in the case of Widman and Pacioli (not Widmann and Paciuoli), leaving the pronunciation to be determined by the reader's linguistic powers. The question is not an easy one to answer when transliteration is necessary, and Professor Cajori would be among the first to agree that he has not seriously attempted to meet it.

A second question that arises relates to the sequence of topics. Shall a book of this kind be written with a respect to general chronological sequence, or shall it rather consider the question of the sequence of subjects? Do we wish to know what was happening in the thirteenth century, for example, or do we wish to have placed before us the growth of such topics as numerical calculation, algebraic symbolism, and the like? It is no adverse criticism of the work in hand to say that the former plan is generally followed in the first part of the book, and the latter in that part which relates to the last three centuries. For the purposes in view, this is possibly the best plan. The question is, however, a difficult one to answer in planning such a book.

A third question relates to source material. How much material of this kind may properly enter into a book for beginners? The question has been raised of late years by many teachers of history and is a serious one. For example, the hieroglyphics on page 23 are not from Ahmes, but does this detract from their value? The cubic on page 228 is not given as it is in Cardan's work; but again, is it therefore less valuable to such as will read this book? It is probable that the question cannot be definitely answered as to how much material of this kind should be given, and probably there is a fair basis for argument as to how exactly it should be reproduced.

In view of the fact that no history of mathematics has ever been written that has not been subjected or could not be subjected to severe criticism with respect to its statements, as witness Eneström's perennial criticisms of Cantor, it will hardly be expected that Professor Cajori's latest edition will, under all the circumstances of its publication, be found to be without flaw. It is, however, interesting to see that the work has been found to be so helpful as to warrant this partial revision. Readers may be pardoned if they join with the author in the regret that a radical revision was impossible for reasons that are apparently commercial.

DAVID EUGENE SMITH.

A Text-Book on Practical Mathematics for Advanced Technical Students. By H. LESLIE MANN, B.Sc., A.R.C.Sc. Longmans, Green and Co. (New York), London, 1915. xi + 487 pp. Price $2.10 net.

As is the case with many texts of this type, the material and its arrangement are determined by some restricted need and the personal preference of the author. This text, we are told, is based on the work of the senior students at the Woolwich Polytechnic, following the line of the author's lectures there to the students in mechanical and electrical engineering during the past nine years. Though based on the work of the seniors, "the book is meant to cover a two or three-years' course" assuming "a knowledge of the fundamental principles of algebra, trigonometry, and mensuration, and the use of logarithms and squared paper."

The arrangement will be likely to impress all teachers of mathematics and most teachers of engineering in America as rather reverse in portions of the book to their accustomed order. In several places we were given the sensation of moving backward into the subject. We cannot see what personal preference led to the scrappy, unsystematic treatment of algebra covering seventeen full pages, eight of which are devoted to approximations and applications of approximate computations, and to giving a cumbersome way of getting the approximate roots of a quadratic equation. Radicals or irrational quantities seem throughout to be replaced by their value to two or three places even when they occur in denominator or divisor. The proper use of significant figures has not received attention.

An elaborate review of trigonometry, including complex quantities and their graphic representation, pages 19–60, is neat and serves its purpose well; yet in passing we observe the order of the cases in the solution of triangles is: I. When three sides are given, II. When two sides and an angle are given, III. When one side and two angles are given. We are far more surprised to find very little of analytic geometry and to have that little introduced in Chapter V by the geometry of three dimensions, pages 61–78, with a touch of plane analytics on the next two pages in defining slope and derivative and no more until Chapters VIII and IX, pages 115–142, which treat also minimum point, point of inflexion, maxima and minima, tangent and normal, angle between a line and a curve, curvature and center of curvature, in order. The treatment of integrations, pages 143–191, and of definite integrals, pages 192–221, is possibly more nearly in the usual order than any other portions of the book. The remainder of the text is devoted to applications of the former subjects. The numerous exercises are very good and seem to come from actual engineering experience.

The last 60 pages are devoted to finite differences and applications. In paragraphs 217 and 220, where the matter of fitting curves to data is treated under the titles "The straight line law" and "General determination of laws," the author seems not to recognize the fact that in curve-fitting the most difficult part by far is the selection of the *type* of equation, but dismisses it by saying "a probable law connecting x and y is known or assumed, and it is necessary to prove that the given tabular values do actually satisfy the law," page 448, and "It is highly probable that one of the three laws

$$"y = a + bx^n \quad (1), \qquad y = b(x + a)^n \quad (2), \qquad y = a + be^{nx} \quad (3)$$

will suit the given values," page 452. It is due the author to state that he has much company in failing to recognize the above fact, and that we are very glad to see these last sixty pages.

We admire the author's aim in making the calculus the center and unity of the book, and his position in claiming that "the calculus cannot be successfully applied to the problems which occur in actual practice until the student has become thoroughly familiar with its underlying principles and methods, and this familiarity can only be obtained by steady practice. It is unfair to a student to give him as a standard form

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

and then expect him to use it as a formula to integrate any function which might resemble it, or by some means reduce to it. This might be working along the line of least resistance, but it is not educational: neither is it to the best interests of the student to whom sound work in differentiation and integration is an absolute necessity."

The text closes with a list of the answers to all the hundreds of exercises and an index. Example 27, page 17, attracted our attention, so we solved it only to find the "true value" .34315 instead of .3429 in the first part, and in the second part .3774 instead of .3781, making the percentage of error 0.318 instead of 0.503 per cent as printed. The difference is probably due to the fact that we rationalized the denominators and divided. The answers should not be carried out so far at any rate.

The publishers have made the book attractive; even the relative dimensions as well as the appearance of the pages are inviting.

CHARLES C. GROVE.

NOTES.

THE April meeting of the Chicago Section of the American Mathematical Society will be devoted in part to a symposium on divergent series and modern theories of summability, the principal speakers being R. D. CARMICHAEL and C. N. MOORE.

AT the annual meeting of the Mathematical Association of America E. V. HUNTINGTON was elected president, D. N.

LEHMER and J. W. YOUNG vice-presidents, and W. D. CAIRNS secretary-treasurer. By amendment of the constitution the office of managing editor was divided, and H. E. SLAUGHT was appointed manager and R. D. CARMICHAEL editor-in-chief of the *Monthly*. The attendance at the meeting numbered 119, including 94 members; 27 persons and institutions were elected to membership. The finances of the association are evidently well managed, the treasurer's report showing a gain of about one hundred dollars for the past year, notwithstanding the cost of the *Monthly*, the subvention paid to the *Annals of Mathematics*, and a small fund set aside for the work of the national committee on mathematical requirements.

THE opening (January) number of volume 19 of the *Transactions of the American Mathematical Society* contains the following papers: "Rational approximations to irrational complex numbers," by L. R. FORD; "On the imaginary roots of a polynomial and the real roots of its derivative," by H. B. MITCHELL; "Relations entre les notions de limite et de distance," by MAURICE FRÉCHET; "On the foundations of the calcul fonctionnel of Fréchet," by A. D. PITCHER and E. W. CHITTENDEN; "Existence theorems for the general real self-adjoint linear system of the second order," by H. J. ETTLINGER; "On boundary value problems in linear differential equations in general analysis," by T. H. HILDEBRANDT; "A fundamental system of formal covariants modulo 2 of the binary cubic," by O. E. GLENN.

THE opening (January) number of volume 40 of the *American Journal of Mathematics* contains: "Flat-sphere geometry. Second paper," by JOHN EIESLAND; "Irrational involutions on algebraic curves," by J. V. DEPORTE; "The set of eight self-associated points in space," by J. R. MUSSELMAN; "Associate minimal surfaces," by J. K. WHITTEMORE; "On integral invariants," by F. W. REED; "Fundamental regions for certain finite groups in S_4," by H. F. PRICE.

THE November number (volume 3, number 11) of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "A necessary and sufficient condition for the existence of a Stieltjes integral," by G. A. BLISS; "Transformations of applicable conjugate nets of curves on surfaces," by L. P. EISENHART; "On bilinear and n-linear functionals," by C. A. FISCHER; "On the deformation of an n-cell," by OSWALD VEBLEN; "A theorem on series of orthogonal functions with an application to Sturm-Liouville series," by G. D. BIRKHOFF.

AT the annual meeting of the British mathematical association held on January 10, 1918, the following papers were read: By W. P. MILNE, "Uses and functions of a school mathematical library"; by S. BRODETSKY, "Nomography"; by G. GOODWILL, "Some suggestions for the presentment of mathematics in closer touch with reality"; by T. P. NUNN (presidential address), "Mathematics and individuality."

AT the meeting of the Edinburgh mathematical society on January 11, the following papers were read: By Miss E. PAIRMAN, "On a difference equation due to Stirling"; by E. T. WHITTAKER, "On Bernoulli's and Fürstenau's methods for the solution of equations"; by E. M. HORSBURGH, "An approximate formula for the length of an arc of a suspended rope."

AT the annual meeting of the Paris academy of sciences, held December 10, 1917, the following prizes in pure and applied mathematics were awarded, in addition to those noted in the February number of the BULLETIN: The Lalande prize to Mr. R. JONCKHEERE for his catalogue of double stars; the Valz prize to Dr. A. SCHAUMASSE, of the University of Nice, for the discovery of the comet 1917 b; the Petit d'Ormoy prize to Professor P. DUHEM, of the University of Bordeaux, for his cosmological theories of the earth; the Binoux prize to Professor G. TEIXEIRA, of the University of Porto, for his mathematical works, and honorable mention to Mr. A. BORDEAUX; the Saintour prize to Professor H. LEBESGUE for his contributions to the principles of the calculus; the de Parville prize to Professor C. J. DE LA VALLÉE POUSSIN for his Course of infinitesimal analysis and Lessons on Lebesgue integrals. The Vaillant, Damoiseau, Fourneyron, Pontecoulant, Pierson-Perrin, and Guzman prizes were not awarded.

All the prizes are to be awarded in 1918 and following years under the usual conditions. The following prize questions are proposed: For the Damoiseau prize (2000 francs) to be awarded in 1920: "To improve in some important points the work of Poincaré and Liapounoff on the figures of equilibrium of a rotating fluid mass subject to newtonian attraction. The

Academy calls especial attention to the question of stability and the study of infinitesimal oscillations about a stable figure." For the Bordin prize (3000 francs) to be awarded in 1919: "In the theory of integrals of total differentials of the third kind and of double integrals relative to an algebraic function of two independent variables, there has been proved the existence of certain integers, whose value is difficult to obtain and may depend on the arithmetic nature of the coefficients of the equation of the surface corresponding to the function. The Academy asks a detailed study of these numbers in some important special cases." For the Grand prize (3000 francs) to be awarded in 1920: "To improve the theory of the functions of one variable that can be represented by trigonometric series of several arguments that are linear functions of that variable."

THE following university and college teachers of mathematics have recently entered the national military service:

Professor J. L. COOLIDGE, of Harvard University, has been commissioned major in the ordnance department and sent to Europe as head of a scientific mission. Mr. W. E. EDINGTON, of the University of Illinois, has joined the research division of the Signal Service Bureau at Leavenworth, Kan. Dr. L. R. FORD, of Harvard University, has joined the coast artillery. Dr. L. M. KELLS, of the University of Illinois, has entered the officers' reserve training camp at Battle Creek, Mich. Mr. F. D. POSEY, of Lebec, Cal., has been made second lieutenant in the national army. Mr. J. L. WALSH, of Harvard University, has joined the naval reserve.

PROFESSOR VITO VOLTERRA, of the University of Rome, has been elected a foreign associate of the Paris academy of sciences.

AT the University of Illinois, Dr. J. E. MCATEE, of William Jewell College, and Mr. L. L. STEIMLEY, of the University of Kansas, have been appointed instructors in mathematics. Dr. J. R. MUSSELMAN has resigned his instructorship to enter statistical work in one of the government departments at Washington. Mr. H. D. FRARY has resigned to become director of the wood-testing plant for aeroplanes at the University of Wisconsin. Mr. A. W. LARSEN has resigned to accept an instructorship at the University of Kansas.

PROFESSOR ELLERY W. DAVIS, dean of the school of arts and sciences and head of the department of mathematics of the University of Nebraska, died February 3 at the age of sixty years. Professor Davis was one of the earliest members of the American Mathematical Society, having entered in 1891.

DR. R. A. HARRIS, of the U. S. coast and geodetic survey and well known for his researches in the theory of the tides, died January 17 at the age of fifty-four years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOUTROUX (E.). Natural law in science and philosophy. Authorized translation by F. Rothwell. New York, Macmillan, 1914. 8vo. 218 pp.

"F. R. S." See THOMPSON (S. P.).

HANCOCK (H.). Theory of maxima and minima. Boston, Ginn, 1917. 8vo. 14 + 193 pp.

KENYON (A. M.) and LOVITT (W. V.). Mathematics for collegiate students of agriculture and general science. New York, Macmillan, 1917. 8 + 357 pp. Cloth. $2.00

LOVITT (W. V.). See KENYON (A. M.).

MAURUS (E. J.). An elementary course in differential equations. Boston, Ginn, 1917. 12mo. 8 + 51 pp. $0.72

ROTHWELL (F.). See BOUTROUX (E.).

THOMPSON (S. P.). ("F. R. S."). Calculus made easy: being a very-simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the differential calculus and the integral calculus. Second edition, enlarged. London, Macmillan, 1917. 12mo. 12 + 265 pp. 2s.

ZONDADARI (E.). Integrazione grafica e studio delle equazioni differenziali ordinarie del primo ordine coi metodi della geometria descrittiva. Milano, Soc. ed. Dante Alighieri (Roma, tip. Nazionale, Bertero), 1917. 8vo. 9 + 112 pp. L. 3.50

II. ELEMENTARY MATHEMATICS.

BARKER (E. H.). Plane trigonometry with tables. Philadelphia, Blakiston, 1917. 8vo. 8 + 172 pp.

CHADSEY (C. E.) and SMITH (J. H.). Efficiency arithmetic. 3 volumes. Boston, Atkinson, Mentzner and Company, 1917. 12mo. Primary: 7 + 280 pp. Intermediate: 6 + 282 pp. Advanced: 5 + 314 pp. $0.40 + 0.40 + 0.45

FOLDBERG (P. T.). Regnebog for Realskoler. 1. Del. København og Kristiania, Gyldendal, 1917. 8vo. 94 pp.

FRIIS-PETERSEN (F.) og JESSEN (J. W. L.). Opgaver i Algebra for Realklassen og det sproglige Gymnasium. København, Gjellerup, 1917. 8vo. 64 pp.

HARPER (G. A.). See NEWELL (M. J.).

JESSEN (J. W. L.). See FRIIS-PETERSEN (F.).

NEWELL (M. J.) and HARPER (G. A.). Plane and solid geometry. Chicago, Row, Peterson and Company, 1917. 12 + 384 pp.

PLATOU (T. H.). Opgaver i Aritmetik og Algebra for Middelskolen. Kristiania, Cammermeyer, 1917. 8vo. 62 pp.

SMITH (J. H.). See CHADSEY (C. E.).

III. APPLIED MATHEMATICS.

BIANCHI (E.). La determinazione del punto da bordo di aeronavi. Parte 1. (Ministero delle armi e munizioni: direzione generale d'aeronautica, istituto centrale aeronautico.) Roma, tip. Unione ed., 1917. 8vo. 158 pp.

——. See DIFFERENZA.

CONTARINO (F.). See DIFFERENZA.

CRAWLEY (E. S.) and EVANS (H. B.). Analytic geometry. Philadelphia, E. S. Crawley, 1918. 14 + 239 pp. $1.60

DEL BUE (A.). Lezione di meccanica generale: cinematica. Città di Castello, Soc. tip. Leonardo da Vinci, 1917. 72 pp. L. 3.00

DIFFERENZA di longitudine fra Roma (M. Mario) e Napoli (Osserv. di Capadimonte) determinate nei mesi di giugno e luglio del 1909 da E. Bianchi, F. Contarino, V. Nobile, e G. Zappa. Parte 1. (R. Commissione geodetica italiana.) Roma, tip. Nazionale, 1917. 4to. 54 pp.

EVANS (H. B.) See CRAWLEY (E. S.).

FARMER (F. M.). Electrical measurements in practice. New York, McGraw-Hill, 1917. 360 pp. $4.00

GRAY (A.). Principles and practice of electrical engineering. 2d edition. New York, McGraw-Hill, 1917. 431 pp. $3.00

HALSEY (F. A.). Handbook for machine designers, shop men and draftsmen. 2d edition. New York, McGraw-Hill, 1917. 561 pp. $5.00

HILLS (R. W.). Machine drawing. New York, McGraw-Hill, 1917. 92 pp. $1.00

JACOBY (H.). Navigation. New York, Macmillan, 1917. $2.25

KENNELLY (A. E.). The application of hyperbolic functions to electrical engineering problems. 2d edition. New York, McGraw-Hill, 1917. 302 pp. $2.50

——. Artificial electric lines. Their theory, mode of construction and uses. New York, McGraw-Hill, 1917. 348 pp. $4.00

LAWS (F. A.). Electrical measurements. New York, McGraw-Hill, 1917. 719 pp. $5.00

LEUTWILER (O. A.). Elements of machine design. New York, McGraw-Hill, 1917. 607 pp. $4.00

MARCOLONGO (R.). Meccanica razionale. Volume 1: Cinematica-statica. Seconda edizione, riveduta ed ampliata. (Manuali Hoepli, N. 352–353.) Milano, Hoepli, 1917. 24mo. 15 + 323 pp. L. 4.50

NOBILE (V.). See DIFFERENZA.

POORMAN (A. P.). Applied mechanics. New York, McGraw-Hill, 1917. 8vo. 12 + 244 pp. $2.00

SCHLAUCH (W. S.). See WENTWORTH (G.).

SCHUSTER (A.) and SHIPLEY (A. E.). Britain's heritage of science. London, Constable, 1917. 16 + 334 pp.

SECRIST (H.). An introduction to statistical methods. A textbook for college students. A manual for statisticians and business executives. New York, Macmillan, 1917. 22 + 482 pp. $2.00

SHIPLEY (A. E.). See SCHUSTER (A.).

SMITH (D. E.). See WENTWORTH (G.).

STEINMETZ (C. P.). Theory and calculation of electrical apparatus. New York, McGraw-Hill, 1917. 480 pp. $4.00

STUART (E. R.). Topographical drawing. New York, McGraw-Hill, 1917. 126 pp. $2.00

WEICK (C. W.). Mechanical drawing problems. New York, McGraw-Hill, 1917. 153 pp. $1.25

WENTWORTH (G.), SMITH (D. E.) and SCHLAUCH (W. S.). Commercial algebra. Book 1. Boston, Ginn, 1917. 12mo. 6 + 266 pp. Cloth. $1.12

ZAPPA (G.). See DIFFERENZA.

L R RY

THE RÔLE OF THE CONCEPT OF INFINITY IN THE WORK OF LUCRETIUS.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society, December 27, 1917.)

No doubt there will one day be written the history of the concept of infinity. If it is to be done by an American scholar, it will probably not be done in the present generation, for the doing of it calls for a kind of composite scholarly preparation—linguistic, historical, philosophical, scientific, and especially mathematical—which our American universities have indeed the machinery but not yet, it seems, the spirit or the purpose or the atmosphere or the temper to provide.

In any adequate historical survey of the rôle of the notion of infinity in our human thinking a consideration of the De Rerum Natura of Lucretius will have to be accorded the position of an important chapter. Most of the many great merits of the work have been long, if not generally nor even widely, recognized. One of its recognized merits is its superb daring and the unsurpassed magnificence of its enterprise; another is its probably unmatched union of literary excellence with scientific spirit and aim; still another, which includes many, being a highly composite merit, is its confident and often acutely argued presentation, sometimes in detail and sometimes in clear and striking outline, of ideas and doctrines that came into their own only in modern science. I refer to such concepts and dogmas as natural law, the atomic constitution of matter, the conservation of mass and of energy, organic evolution, spontaneous or chance variation of life forms, struggle for existence, survival of the fit, and sensation as the ultimate basis of knowledge and the ultimate test of reality, not to mention other equally brilliant anticipations of modern scientific thought.

In attempts to appraise the work of Lucretius his employment of the notion of infinity is commonly indicated, but only more or less incidentally. For example, in Masson's large volume, Lucretius, Epicurean and Poet, the term infinite has only a subordinate place in the index of important terms; in

Munro's very extensive Notes the term receives but scant attention; and it receives even less in the Notes found in Cyril Bailey's recent and deservedly much praised English translation of the poem. What is missed in such appreciations and commentaries and what I wish to signalize here is the fact that the concept of infinity—of infinite multitude and infinite magnitude—is not merely one among the many ideas, but is indeed the dominant idea, in the work of Lucretius. A critical examination of the work cannot fail to discover that in the author's judgment the concept in question was at once the most powerful of his instruments and the one most obviously indispensable to the success of his great undertaking. That undertaking was a pretty large contract, being nothing less than the endeavor to show forth a method by which it would be possible to explain, or to account for, all phenomena (whether mental or not) without having to resort to the hypothesis of divine intervention.

This is not the place to give a detailed account of the Lucretian principles and procedure. For the purpose of this note it is sufficient to point out that among the fundamental propositions there are three major propositions and that these owe their efficacy and their dominance of the entire discourse to the fact of their postulating the existence of infinite multitude and infinite magnitude. These propositions are that the universe of space is a region or room of infinite extent; that time is an infinite duration composed of an infinite past and an infinite future; and that the matter in the universe is composed of an infinite multitude of absolutely solid (non-porous) and non-decomposable atoms or "seeds of things" always moving hither and thither in an infinite variety of ways and ever so distributed throughout the whole of space that of all spheres none but such as are microscopically minute could at any given instant fail to enclose one or more of the "seeds." Without these postulated infinites explanation of the phenomena of the world was, in the belief of Lucretius, impossible; with them, supplemented by certain other postulates, such explanation was possible. Though the mentioned postulates were not in themselves sufficient, they were regarded as conspicuously necessary. In the view of Lucretius cosmic history was an eternal drama enacted by an infinitude of unoriginated and indestructible elements operating upon an infinite stage. The drama was not to be understood except by help of the concept of infinity; and so the De Rerum Natura may be not unjustly said to be a kind of poetic celebration of what the author deemed to be the scientific efficacy of that concept.

What did Lucretius mean by infinity? What did he mean by an infinite multitude and by an infinite magnitude? No formal definition of any of these terms is to be found in his work. But it is perfectly clear that he conceived an infinite multitude of elements to be a multitude which could not be exhausted by removing from it one element at a time but which could be thought as arranged in an endless succession of elements. In other words, an infinite multitude signified what we now describe as a denumerably infinite multitude. There is no hint at any conception of any higher order of infinity.

It is noteworthy that, with the possible exception of time, the fundamental infinites of Lucretius were not mere variables capable of increase beyond any prescribed finite amount; on the contrary, they were, like the infinites of Cantor, constant or static affairs; but, unlike the Cantor infinites, those of Lucretius were composed of actual, or concrete, things and not abstract ones like points, for example, or pure numbers; thus the Lucretian infinitude of atoms were material particles and they all existed at once.

Was Lucretius aware of the fact that an infinite multitude, as conceived by him, contained parts or submultitudes equivalent, as we now say, to the whole? He was not only aware of it but he repeatedly employed this characteristic property of infinity effectively and correctly. This rather astonishing fact is sufficiently interesting to justify citation of one or two of the numerous passages supporting its assertion. If we bear in mind that one of the Lucretian infinites was the succession of time units (days or generations or other stretches) beginning at any given instant and together composing what is called the future, the following famous passage makes it perfectly clear that, according to its author, the removal of any finite multitude of elements from an infinite multitude of them leaves a remainder, a part, exactly equal or equivalent to the whole:

"Nor by prolonging life do we take one tittle from the time past in death nor can we fret anything away, whereby we may haply be a less long time in the condition of the dead. Therefore you may complete as many generations as you please;

none the less however will that everlasting death await you; and for no less long a time will he be no more in being, who beginning with today has ended his life, than the man who has died many months and years ago." *

Lucretius, as already said, postulated the existence of an infinity of atoms. These were not all of them identical in shape or in size but in these respects presented a finite number of varieties. Each variety, it was further assumed, constituted an infinite multitude.† These infinites were not distinguished as such; in respect of multiplicity, though not in respect of their physical functions, they were equivalent to one another and to the whole multitude constituted by their elements.

It is sufficiently evident that the poet's conception of infinite multitude was identical with that now employed by mathematicians. Careful scrutiny of the poem will discover that the same may be said of the author's conception of infinite magnitude. Formal definition of the notion is not present. We are told, however, that in respect of size, all the atoms are between a finite upper bound and a finite lower bound; we are told that the sum of any finite number of atoms is finite; we are told that the sum of all atoms of a given form is infinite and that, therefore, the number of them must be infinite. It is thus evident that for Lucretius an infinite magnitude is a magnitude greater than the sum of any finite number of finite quantities none of which surpasses, in respect of parvitude, a finite size.

Formation or possession of ideas is one thing; logically correct handling of them in argumentation is quite another. In his use of the ideas in question Lucretius was sometimes right and sometimes wrong. Of right use some examples have already been given and it would be easy to cite others. As an instance of erroneous use the following passage (as correctly translated by Munro, page 15) may be cited on account of its great interest:

"Again unless there shall be a least, the very smallest bodies will consist of infinite parts, inasmuch as half of a half will always have a half and nothing will set bounds to the division. Therefore between the sum of things and the least of things

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* Munro's Lucretius, 4th ed., p. 83.

† Cf. Munro, pp. 39, 40, 41, or Bailey's Lucretius On the Nature of Things, pp. 76, 83, 84.

what difference will there be? There will be no distinction at all; for how absolutely infinite soever the whole sum is, yet the things which are smallest will equally consist of infinite parts."

The significance of the passage and the nature of the error contained in it will be clearer if we note that it is a portion of the argument by which the author seeks to establish his contention that among the parts composing an atom (which is by assumption the smallest material particle capable of existing spatially separate from all other particles) there is a least part, by which is meant a part so small that none of the parts is smaller. Paraphrased in modern terms this portion of the argument would run about as follows: if among the parts composing an atom and being such that no two of them have points in common save points of a common surface there be no least part, then the atom consists of an infinite number of parts; the number of atoms in the universe is infinite; these two infinite multitudes of finite portions of matter are equivalent; the sum of the elements of the latter multitude is an infinite magnitude; such, too, is the sum of the elements of the former multitude; but this sum is the atom itself; hence, unless there be a least part among the parts of an atom, an atom is an infinite magnitude and as such is no less than the sum of all matter. The error is not due to a wrong conception of infinity, whether of multitude or of magnitude, but is obviously due solely to the tacit assumption of the false proposition that the sum of the elements of any infinite multitude of finite elements is infinite. An obvious moral is that a little knowledge of the convergence of series would greatly improve the philosophy of poets and the science of philosophers.

It is astonishing that the mentioned fallacy occurs in immediate conscious connection with a line seeming to refute it: "the half of the half will always have a half." What is the explanation? It is not to be found in any supposition of stupidity or of momentary nodding. It is to be found in the author's purpose and point of view. He was exclusively concerned with natural phenomena, with what he deemed to be existing entities. Regarding the series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$, he would have said in effect: "Composed of man-made symbols like words, it is not nor ever can be endless; to speak of the sum of a non-existing endless series is meaningless; moreover, even if we supposed the series to be endless, to be summable, and to have 1 for its sum, this 1 would be neither finite nor infinite, for it would not be a magnitude, inasmuch as the summands are themselves not magnitudes but are merely empty abstract symbols; if 1 be said to be a magnitude, in the sense of representing a magnitude, then, if the magnitude 1 be composed of two equal magnitudes, $\frac{1}{2}$ will be a magnitude in the same sense, and so on for the other symbols; if all the symbols be magnitudes in the indicated sense, the summation of the abstract series will be the summation of an endless series of magnitudes; otherwise not; and now what I have contended in my book is that, if the magnitude 1 be finite, not more than a relatively few of the symbols of the series can be magnitudes, and this contention, denying the infinite divisibility of finite magnitude, is based on grounds other than that advanced in the above-quoted passage from my argument." This is not the place to recount, much less to estimate, those "other" grounds. It must, however, be said, in passing, that one of them is in point of kind almost perfectly indicated by the following words of Clerk Maxwell:* "What we assert is that after we have divided a body into a certain finite number of constituent parts called molecules, then any further division of these molecules will deprive them of the properties which give rise to the phenomena observed in the substance."

The classic form of the false thesis tacitly invoked by Lucretius in the above-quoted passage to fortify his "other" grounds for denying the infinite divisibility of a finite magnitude is exceedingly vague: all infinities are equal. It has had an age-long and world-wide vogue. Thus Kanadi, an old Hindoo author, employs the thesis to prove that, if every body be infinitely divisible, there can be "no difference of magnitude between a mustard seed and a mountain." † In this connection mathematicians, especially those who may be interested in the history of the notion of infinity, will be glad to have their attention called to a little-known letter‡ of Newton dealing with the matter. The letter, which is addressed to Richard Bentley, is very interesting on several accounts: it points out the vagueness and the falsity of the above-mentioned thesis, which Dr. Bentley had assumed to be true; it repeatedly employs the term infinite in a sense not less vague and indeterminate; it virtually asserts that, if two infinite magnitudes be equal, the addition of any finite magnitude to either of them will destroy the equality.

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* Theory of Heat, p. 285.

† See Daubeny's Introduction to the Atomic Theory, p. 5.

‡ See The Works of Richard Bentley, vol. III, p. 207.

In closing this note I desire to guard against the danger of leaving a false impression. The mere correctness of the Lucretian concept of infinity by no means accounts for the immense rôle of the concept in the author's work. The secret lies in the fact that the concept so powerfully stimulated the imagination of a great thinker and poet as to cause him to express and to preserve in immortal form a body of ideas which he had acquired from the then still extant works of Epicurus and which after the long lapse of centuries are found to be among the most fruitful scientific ideas of our time.

COLUMBIA UNIVERSITY,
December, 1917.

ON THE INVARIANT NET OF CUBICS IN THE STEINERIAN TRANSFORMATION.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society September 4, 1917.)

1. BY Steinerian transformation* we understand an involutorial quadratic Cremona transformation, defined as the one-to-one correspondence between the points of a plane (with the exception of the points of a certain trilateral) and the points of concurrence of their polars with respect to the conics of a pencil. If we use the base points A_1, A_2, A_3, E of the pencil as the vertices and the unit point of a system of projective coordinates, the Steinerian transformation may easily be established in the form

$$\begin{aligned} \rho x_1' &= x_1(x_2 + x_3 - x_1), \\ \rho x_2' &= x_2(x_3 + x_1 - x_2), \qquad (1) \\ \rho x_3' &= x_3(x_1 + x_2 - x_3). \end{aligned}$$

The base points $A_1(1, 0, 0)$; $A_2(0, 1, 0)$; $A_3(0, 0, 1)$; $E(1, 1,$

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[*] See *Annals of Mathematics*, vol. 14 (1912), pp. 57–71.

1) of the pencil are invariant under the transformation, while to the diagonal points $E_1(0, 1, 1)$; $E_2(1, 0, 1)$; $E_3(1, 1, 0)$ correspond the opposite diagonals of the diagonal triangle of the base point quadrangle. The net of cubics through the seven points A_1, A_2, A_3, E, E_1, E_2, E_3 may be represented in the form

$$(2) \quad a_1x_2x_3(x_2 - x_3) + a_2x_3x_1(x_3 - x_1) + a_3x_1x_2(x_1 - x_2) = 0,$$

and is invariant under the transformation (1). It is not difficult to establish the

THEOREM: *For every set of values* $(a_1 : a_2 : a_3)$ *equation* (2) *represents a cubic for which the coordinate triangle* $A_1A_2A_3$ *and the unit point* E *form a Steinerian quadruple, such that the tangents to the cubic at these points meet in a point* P *of the cubic whose coordinates are precisely* a_1, a_2, a_3.

Conversely, when from a point P *of an elliptic cubic we draw the four tangents to the cubic with the points of tangency* A_1, A_2, A_3, E, *and we choose these as the vertices and unit point of a projective coordinate triangle, in which the coordinates of* P *are* a_1, a_2, a_3, *then the equation of the cubic may be written in the form* (2).

2. It is known that on every straight line l in a general position there are two points P and P' only which correspond to each other in the Steinerian transformation. They are obtained as the points of intersection of l and the corresponding conic K through the diagonal points, or also as the double points of the involution cut out on l by the conics of the pencil through the base points. The cubic may be generated by the pencil of lines through P and the projective pencil of corresponding conics through E_1, E_2, E_3, and P'. It may also be considered as the locus of pairs of corresponding points on the rays of the pencil through P. For the sake of brevity the point of concurrence of the tangents to a cubic at the points of an inscribed Steinerian quadruple shall be called an S-point of the cubic. Two cubics C_P and C_Q of the net intersect in two points X and X'. As both are invariant under the transformation, the corresponding point of X must lie on C_P as well as on C_Q, and must therefore coincide with X'. Hence the

THEOREM: *Two cubics of the net associated with the complete quadrangle of a Steinerian transformation always intersect in two corresponding points.*

All cubics of the net have, of course, the seven fixed points of the complete quadrangle in common.

3. Let $P(a_1, a_2, a_3)$ and $Q(b_1, b_2, b_3)$ be the S-points of two cubics C_P and C_Q of the net, so that the equation of the pencil of cubics C_λ determined by C_P and C_Q may be written in the form

$$(3)\quad (a_1 + \lambda b_1)x_2x_3(x_2 - x_3) + (a_2 + \lambda b_2)x_3x_1(x_3 - x_1) + (a_3 + \lambda b_3)x_1x_2(x_1 - x_2) = 0.$$

Obviously, the S-point of every cubic of this pencil with the parameter λ has the coordinates

$$a_1 + \lambda b_1, \qquad a_2 + \lambda b_2, \qquad a_3 + \lambda b_3,$$

and lies on the line joining the S-points P and Q of C_P and C_Q.

As the equation of every pencil of the net may be written in the form (3) we may state the

THEOREM: *The S-points of the cubics of a pencil within the net of cubics associated with a Steinerian quadruple are collinear, and form a point set projective with the pencil of cubics.*

Let l be the line of collinearity of the S-points of the pencil of cubics, and S_λ the S-point on l associated with the cubic C_λ of the pencil. The cubic C_λ will pass through the two corresponding points P and P' of the Steinerian transformation which lie on l. As there are two and only two such points on l, it is clear that all cubics of the pencil must pass through these points. We shall call the locus l of the S-points of the pencil the S-line of the pencil of cubics C_λ.

Consider now the pencil of cubics C_λ, and join every S-point on the S-line l to a fixed point S not on l. In this manner we get the pencil of lines through S with the equation (s_1, s_2, s_3 being the coordinates of S)

$$(4)\quad (s_2a_3 - s_3a_2)x_1 + (s_3a_1 - s_1a_3)x_2 + (s_1a_2 - s_2a_1)x_3 + \lambda\{(s_2b_3 - s_3b_2)x_1 + (s_3b_1 - s_1b_3)x_2 + (s_1b_2 - s_2b_1)x_3\} = 0,$$

which is projective with the set of S-points, and consequently with the pencil of cubics (3). These two projective pencils generate a quartic whose equation is found as

$$
(5)\quad \{(a_2b_3 - a_3b_2)x_1 + (a_3b_1 - a_1b_3)x_2 + (a_1b_2 - a_2b_1)x_3\} \times \{s_1x_2x_3(x_2 - x_3) + s_1x_3x_1(x_3 - x_1) + s_3x_1x_2(x_1 - x_2)\} = 0.
$$

Clearly, the quartic degenerates into the S-line of the pencil of cubics, and into a cubic having (s_1, s_2, s_3) as the S-point. Hence the

THEOREM: *Any cubic of the net with a given S-point may be generated by any pencil of cubics within the net, not containing the given cubic, and the projective pencil of lines joining the S-point of the given cubic to the S-points of the cubics of the pencil.*

Consider next two pencils of cubics C_λ with the S-line l, and C_μ with the S-line m, and a point S, not on l or m. Draw any line g through S, cutting l and m in S_λ and S_μ, and construct the cubics C_λ and C_μ having S_λ and S_μ as S-points.

They both pass through the two fixed points P and P' on g corresponding to each other in the Steinerian transformation. But P and P' also lie on the cubic C_S associated with S as an S-point. For a variable g through S, S_λ and S_μ describe two perspective point sets on l and m which are projective with the pencils of cubics C_λ and C_μ. These pencils are therefore themselves projective, and generate the cubic C_S. Hence the

THEOREM: *Every cubic of the net associated with a Steinerian transformation may be generated in an infinite number of ways by projective pencils of cubics of the same net.*

UNIVERSITY OF ILLINOIS.

SOME THEOREMS OF COMPARISON AND OSCILLATION.

BY PROFESSOR TOMLINSON FORT.

(Read before the American Mathematical Society September 4, 1917.)

THEOREM I: *Given*

$$
(1)\qquad \frac{d}{dx}(K_1(x)y_1') + G_1(x)y_1 = 0
$$

and

$$
(2)\qquad \frac{d}{dx}(K_2(x)y_2') + G_2(x)y_2 = 0,
$$

where K_1, K_2, G_1 *and* G_2 *are real and absolutely integrable in the Riemann sense over the interval* $a \leqq x \leqq b$, *and where* $0 < K_2 \leqq K_1$ *and* $0 < G_1 < G_2$.

Let y_1 *and* y_2 *be particular, not identically vanishing, solutions of* (1) *and* (2) *respectively such that either*

Case I $$y_1(a) = 0$$

or

Case II $$\frac{K_1(a)y_1'(a)}{y_1(a)} \geqq \frac{K_2(a)y_2'(a)}{y_2(a)}.$$

Let $V_i(x) = \alpha_1(x)K_i(x)y_i'(x) + \alpha_2(x)y_i(x)$, $i = 1, 2$, *where* $\alpha_1(x)$ *and* $\alpha_2(x)$ *are real functions defined and continuous when* $a \leqq x \leqq b$ *and such that* $\alpha_1(x) \neq 0$ *at any point and* $\alpha_2(x)/\alpha_1(x)$ *never decreases as* x *increases. Assume that* $y_1(a) \geqq 0$, $y_2(a) \geqq 0$ *and* $V_1(a) \cdot V_2(a) \geqq 0$ *and that, if* $V_2(a) = 0$, $V_1(a) = 0$ *also.*

If $V_1(x)$ *has a root at a point* c, *where* $a < c \leqq b$, *then* $V_2(x)$ *has a root at a point* d, *where* $a < d < c$.

Without loss of generality we assume $\alpha_1 > 0$. If $\alpha_1 < 0$ we simply change the signs of both α_1 and α_2.

Case I. Assume $y_1'(a) > 0$ and, if $y_2(a) = 0$, $y_2'(a) > 0$ also, in which again there is no loss of generality. Let $l = a + \delta$ where $\delta > 0$ is very small. Then

$$K_1y_1'y_2 - K_2y_2'y_1]_a^l = \int_a^l (G_2 - G_1)y_1y_2dx + \int_a^l (K_1 - K_2)y_1'y_2'dx.$$

In case $y_2(a) = 0$ and δ is sufficiently small, $y_1y_2 > 0$ when $a < x \leqq l$ and $y_1'y_2' > 0$ when $a \leqq x \leqq l$. Consequently

$$\frac{K_1(l)y_1'(l)}{y_1(l)} > \frac{K_2(l)y_2'(l)}{y_2(l)}.$$

If $y_2(a) \neq 0$ this result is immediate from the continuity of y_1, y_1', y_2 and y_2'. Replace the point a by the point l and Case I reduces to Case II.

Case II. Assume that $V_2(x)$ does not have a root on the interval $a < x < c$ and assume that c is the first root of $V_1(x)$ greater than a. Choose $y_1(a) = y_2(a)$.

(a) Suppose $V_2(a) > 0$ and consequently $V_1(a) > 0$.

$$y_1^2\left(\frac{K_1y_1'}{y_1}-\frac{K_2y_2'}{y_2}\right)\Bigg]_a^c=\int_a^c(G_2-G_1)y_2^2dx$$

$$+\int_a^c(K_1-K_2)y_2'^2dx+\int_a^c K_2\left(\frac{y_1'y_2-y_1y_2'}{y_2}\right)^2dx.^*$$

y_1 and y_2 are both positive over the interval of integration since

$$\frac{K_1(x)y_1'(x)}{y_1(x)}$$

becomes negatively infinite as x approaches c, and similarly

$$\frac{K_2(x)y_2'(x)}{y_2(x)}$$

becomes negatively infinite as x approaches the first root of y_2 greater than a. Consequently

$$K_1(c)y_1'(c)y_2(c)-K_2(c)y_2'(c)y_1(c)>0.$$

Combining this with $V_1(c)=0$, we get $y_1(c)V_2(c)<0$, which is impossible.

(*b*) $V_2(a)\leqq 0$ and $V_1(a)\leqq 0$.

$$\frac{K_i(x)y_i'(x)}{y_i(x)} \qquad (i=1,2)$$

decreases as x increases, as is proved by differentiation. Consequently $y_i(x)=0$ to the right of a necessarily before $V_i(x)=0$. Suppose $y_1=0$ at g, where $g<c$; then, as is well known, $y_2=0$ at h where $h<g$. There is only one such point less than c, for by hypothesis $V_2(x)$ has no root when $a<x<c$. Take a point $l=g+\delta$, where δ is small. At this point, with δ small enough, we have the same situation as at a under (*a*) if y_2 is replaced by y_2 multiplied by such a constant as to cause $y_2(l)$ to equal $y_1(l)$ and then y_1 and y_2 replaced by $-y_1$ and $-y_2$ respectively.

THEOREM II. *Let* y_1 *and* y_2 *be any two not identically vanishing solutions of* (1) *and* (2) *respectively and let*

$$V_i(x)=\alpha_1(x)K_i(x)y_i'(x)+\alpha_2(x)y_i(x) \qquad (i=1,2).$$

Between any two successive roots of $V_1(x)$ *on the interval* $a\leqq x\leqq b$ *lies at least one root of* $V_2(x)$.

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[*] This is a well known identity. It can easily be verified.

Suppose two successive roots of $V_1(x)$ at c_1 and c_2 respectively. Without loss of generality we assume $y_1(c_1) > 0$ and $y_2(c_1) \geqq 0$; if $y_2(c_1) = 0$, $y_2'(c_1) > 0$. Since in addition $K_1(x)y_1'(x)/y_1(x)$ decreases in x and $\alpha_1 > 0$, $V_1(x) < 0$ when $c_1 < x < c_2$.

Case I. $V_2(x) < 0$ when $c_1 < x < c_2$. Conditions at a of Theorem I, Case II b, are fulfilled at c_1.

Case II. $V_2(x) > 0$ when $c_1 < x < c_2$, $y_1 = 0$ at g, where $c_1 < g < c_2$, since $K_1(x)y_1'(x)/y_1(x)$ always decreases. At g we have the conditions of Theorem I, Case I, y_1 being replaced by $-y_1$.

Suppose next that we have the equation

$$\frac{d}{dx}(K(x, \lambda)y') + G(x, \lambda)y(x) = 0, \tag{3}$$

where $K(x, \lambda)$ and $G(x, \lambda)$ are real and continuous in λ, $-\infty < \lambda < \infty$, for any x, and with respect to x absolutely integrable in the Riemann sense from a to b, and where $K > 0$ never decreases as λ increases, and G actually increases from values less than or equal to zero to values as large as desired and is such that, for a particular λ, $G \equiv 0$ in x or does not vanish at all.

Consider (3) subject to the boundary conditions

$$\begin{aligned} \beta_1 K(a, \lambda)y'(a) + \beta_2 y(a) &= 0, \\ \alpha_1 K(b, \lambda)y'(b) + \alpha_2 y(b) &= 0, \end{aligned} \tag{4}$$

where α_1, α_2, β_1 and β_2 are real constants, $\alpha_1 > 0$ and β_1 and β_2 are not both zero.

Let $V(x) = \alpha_1 K(x, \lambda)y'(x) + \alpha_2 y(x)$. It is to kept in mind that from now on α_1 and α_2 are constants.

Define $K(x, \lambda)$ and $G(x, \lambda)$, when $x > b$, thus: $K(x, \lambda) = K(x, b)$ and $G(x, \lambda) = G(x, b)$.

Let y, not identically equal to zero, satisfy the first of equations (4). Then notice that V and V' are continuous in λ and that they do not simultaneously vanish when $G \geqq 0$.

Moreover when $G \equiv 0$ and $V(x) \not\equiv 0$, $V(x)$ has at most one root on the interval $a \leqq x \leqq b$, as is readily proved by solving the equation. Consequently in this case when $G > 0$ but sufficiently small, $V(x)$ has only one root on the interval $a \leqq x \leqq b$. $V \equiv 0$ is only possible when $V = 0$ reduces to $y' = 0$. For values of $G > 0$ but sufficiently small in this instance, $V \neq 0$ at all on the interval $a < x \leqq b$, since $K(x, \lambda)y'(x)/y(x)$

decreases as λ increases. Moreover when λ is sufficiently large, as is well known, y and consequently V have as many roots as desired when $a < x < b$. Using these facts and Theorems I and II we now readily prove the following:

THEOREM III. *There exist unique values,* $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, *such that when* $\lambda = \lambda_j$, $j = 1, 2, 3, \cdots$, *a solution* y *of* (3) *exists satisfying* (4) *such that* $V(x)$ *has exactly* j *roots on the interval* $a < x < b$.

We can extend this to $j = 0$ if, when $G = 0$, V has no root on the interval $a < x < b$.

THE UNIVERSITY OF ALABAMA,
September, 1917.

NOTE ON INFINITE SYSTEMS OF LINEAR EQUATIONS.

BY DR. W. L. HART.

(Read before the American Mathematical Society April 28, 1917.)

IN considering infinite systems of linear equations

$$(1) \qquad \sum_{j=1}^{\infty} a_{ij}x_j = x_i' \qquad (i = 1, 2, \cdots),$$

particular interest is attached to those whose solutions preserve the properties of the solutions of a set of n linear equations in n variables. It is known* that the system (1) possesses this property if $a_{ij} = d_{ij} - b_{ij}$ $(d_{ii} = 1; d_{ij} = 0, i \neq j)$, where the infinite matrix $B = (b_{ij})_{i,\, j=1,\, 2,\, \cdots}$ is completely continuous.† The discussion of Riess deals only with the special case $a_{ij} = d_{ij} - b_{ij}$, but it is easily found that his proof holds for the more general case stated below. The proof of the theorem of this note is not given since it differs only in minor details from the proof of the theorem given by Riess.

It will be said that a matrix A_1 is a sub-matrix of the matrix

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* Cf. F. Riess, Equations Linéaires, p. 94.

† Cf. F. Riess, loc. cit.

$A = (a_{ij})_{i,\,j=1,\,2,\,\cdots}$ in case it is obtained from A by striking out certain k rows and k columns (k finite).

Let the system (1) be taken in the field of complex numbers and suppose that the sets of numbers $x = (x_1, x_2, \cdots)$ and $x' = (x_1', x_2', \cdots)$ represent points in Hilbert space; that is, $\sum_{i=1}^{\infty} |x_i|^2$ and $\sum_{i=1}^{\infty} |x_i'|^2$ converge. The main result of the present note is

THEOREM I. *Let the matrix* $A = (a_{ij})_{i,\,j=1,\,2,\,\cdots}$ *be limited in the sense of Hilbert.** *Suppose there exists a sub-matrix* A_1 *possessing a unique limited reciprocal* A_1^{-1}. *Then, the system of linear equations* (1) *preserves the characteristics of a system of n linear equations in n unknowns.*

That is, either (*a*) for every x' system (1) possesses a unique solution, or (*b*) there is an integer $h > 0$ such that if x' is a point in Hilbert space satisfying certain h linear relations, there exists a solution of (1) depending linearly on h parameters.

With the help of certain details necessary for the proof of Theorem I, there is easily proved

THEOREM II. *If the matrix A satisfies the hypotheses of Theorem I and if the homogeneous system*

$$\sum_{j=1}^{\infty} a_{ij}x_j = 0 \qquad (i = 1, 2, \cdots) \tag{2}$$

has no solution in Hilbert space except $x = 0$, *then A possesses a unique limited reciprocal matrix.*

The proof is simple because the reciprocal matrix of the theorem is actually constructed during the proof of Theorem I.

For the system (1) with $a_{ij} = d_{ij} - b_{ij}$ where $B = (b_{ij})$ is completely continuous, the hypothesis of Theorem I is always satisfied by taking, as the sub-matrix A_1, the infinite matrix obtained by striking out the first n rows and n columns where n is taken sufficiently large. Such a matrix is known† to have a unique limited reciprocal matrix.

83RD ARTILLERY,
FORT D. A. RUSSELL, WYOMING,
November 10, 1917.

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* Hellinger and Toeplitz, *Math. Annalen*, vol. 69 (1910), p. 299.

† Riess, loc. cit., p. 98.

AN UNDERVALUED KIRKMAN PAPER.

BY PROFESSOR LOUISE D. CUMMINGS.

(Read before the American Mathematical Society October 27, 1917.)

THE purpose of this note is to emphasize the importance in the theory of triad systems of a Kirkman* paper, which appears to have been overlooked by all writers on this subject up to the present time.

A short explanation of the symbols employed in the paper is necessary. The symbol $Q_{x,y,z}$ denotes the greatest number of combinations that can be made with x elements, y at a time, so that no combination of z elements shall be twice employed; for brevity $Q_{x,3,2}$ is replaced by Q_x. The symbol V_x denotes the number of pairs possible with x elements that are excluded from Q_x. The symbol q_x denotes the number of triads formed with x elements, in which no duad is twice employed, q_x being not necessarily a maximum; v_x is the number of duads possible with x elements not employed in q_x. Four pairs such as 12, 23, 34, 41 forming a closed circle are denoted by the symbol C_4.

The object of Kirkman's paper is to determine the value of Q_x, and to establish the following theorems:

$$Q_x = \frac{x(x-1)}{6} - \frac{V_x}{3},$$

where

$$V_x = \frac{x}{2} + 3k + 1 \quad \text{if } x = 6n - 2;$$

$$V_x = 6k + 4 \quad \text{`` } x = 6n - 1;$$

$$V_x = \frac{x}{2} \quad \text{`` } x = 6n,\ 6n + 2;$$

$$V_x = 0 \quad \text{`` } x = 6n + 1,\ 6n + 3;$$

$$2^m(2k + 1) = n; \quad n, m, x, k \text{ are integers} \geq 0.$$

The case of special interest to us is that in which $V_x = 0$, for only under this condition does a triad system on x elements exist. Kirkman proves the two following propositions:

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* *Cambridge and Dublin Math. Journal*, vol. 2 (1847).

I. If $x = 2n + 1$ and $V_x = 0$, then $V_{2x+1} = 0$, and $v_{2x-1} = C_{2x-2}$; that is, if x is odd and a triad system on x elements exists, then a triad system on $2x + 1$ elements may be constructed which contains a circle of $2x - 2$ pairs chosen from $2x - 1$ elements. This theorem gives the construction for one type of the headed triad systems $\Delta_7, \Delta_{15}, \Delta_{19}, \Delta_{27}, \cdots$.

II. If $x = 2n$ and $v_{x+1} = C_x$, then $V_{2x+1} = 0$, and

$$v_{2x-1} = C_{2x-2};$$

that is, if x is even and if from $x + 1$ elements a partial triad system can be formed such that the unemployed pairs form a closed circle of x pairs, then a triad system on $2x + 1$ elements may be constructed which contains a circle of $2x - 2$ pairs chosen from $2x - 1$ elements. This theorem gives the construction for one type of triad systems $\Delta_9, \Delta_{13}, \Delta_{25}, \cdots$.

These two propositions enable Kirkman to prove that a triad system exists for every value of x of the form $6n + 1$ or $6n + 3$. This result has, in general, been attributed to Reiss,* but, as the Reiss paper was not published until 1859, twelve years after that of Kirkman, the credit properly belongs to Kirkman. A comparison of the Reiss paper with that of Kirkman reveals a remarkable identity of method in the work of these two men. The construction given by Reiss for a triad system on $2x + 1$ elements, x being an odd number, is a perfect duplication of the work of Kirkman. The Reiss construction for a triad system on $2x + 1$ elements, x being an even number, may seem at first sight to differ from that given by Kirkman, but investigation reveals the fact that the two methods are essentially the same and always furnish congruent triad systems. This curious duplication by Reiss of the Kirkman work gives rise to the question as to whether Reiss had a knowledge of this Kirkman paper, or whether the problem was of such a type that two investigators would inevitably have followed the same line of argument.

In Crelle's *Journal für Mathematik*, vol. 45 (1853), Steiner proposes some problems in combinations now generally known as the Steiner combination problems. The first of these problems is only a special case of the Kirkman symbol $Q_{x, y, z}$, and the remaining problems are slight modifications of the Kirkman problem. Since Kirkman published the general problem ($Q_{x,y,z}$) as early as 1844 in the *Ladies' and Gentlemen's Diary*, and, by 1850, had already contributed two papers on the subject to the *Cambridge and Dublin Mathematical Journal*, the *Kirkman* combination problem would seem to be a more appropriate name for the so-called Steiner problem (*a*). Moreover, the Steiner problems (*b*), (*c*), (*d*), (*e*), should be designated as the Steiner modifications of the Kirkman combination problem.

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* M. Reiss, "Ueber eine Steinersche kombinatorische Aufgabe." *Crelle's Journal*, vol. 56 (1859), p. 326.

Netto makes no mention of this Kirkman paper either in his Vorlesungen or in his article on Tripelsysteme in the Encyklopädie der Mathematischen Wissenschaften. In the Encyklopädie article, Netto refers only to two papers by Kirkman, one, in volume 7 (1852), and the other, in volume 8 (1853) of the *Cambridge and Dublin Mathematical Journal*. Now, volume 7 contains no paper by Kirkman on any subject, and the paper in volume 8 is Kirkman's third paper on the problem $Q_{x,y,z}$. That Netto in preparing an article for the Encyklopädie should have overlooked entirely Kirkman's two earlier papers, published in the *Journal* in volumes 2 and 5 respectively, seems the more astounding, since he had at his disposal, not only a very good library at Giessen, but also all of the great mathematical libraries of Germany. The comparative rarity of the *Cambridge and Dublin Journal* is the apparent excuse for the ignorance of Kirkman's work and the exaltation of Steiner and Reiss, but this is a weak argument for Netto when he undertakes an Encyklopädie article.

Another point worthy of attention in the Kirkman paper is represented in the notation $v_{2x-1} = C_{2x-2}$. Interpreted in terms of the special case of fifteen elements, Kirkman shows that a triad system constructed by his method contains a closed circle of twelve pairs involving thirteen elements. Here, then, is the first example of what Professor F. N. Cole, in his work on triad systems, has designated as the dodekad interlacing of the triads. Investigation shows that this Kirkman system contains 56 of these dodekad interlacings, and that among the 80 non-congruent triad systems on fifteen elements enumerated by Professor Cole, 78 contain interlacings of this type.

The remaining propositions in this Kirkman paper concern the determination of Q_x when a triad system on x elements does not exist, and, therefore, are steps towards the solution of the hitherto so-called Steiner problem (*a*).

The school-girl problem is merely an example which originated in the development of this paper on combinations, and Kirkman justly complained of the almost total eclipse of this paper in the wide popular interest aroused by the school-girl problem. The eclipse appears to have continued up to the present day, since no mention is made of this Kirkman paper by Steiner, Reiss, Netto, or by any of the recent writers on triad systems.

VASSAR COLLEGE,
October, 1917.

PIERRE LAURENT WANTZEL.

BY PROFESSOR FLORIAN CAJORI.

(Read before the American Mathematical Society September 4, 1917.)

EVERY one knows that one of the noted proofs of the impossibility of an algebraic solution of the general quintic equation is due to Wantzel. Nevertheless histories of mathematics and biographical dictionaries are silent regarding his life. The eleven papers listed in Poggendorff's Handwörterbuch as due to "Pierre Laurent Wantzell" do not include the proof in question, and a query is raised in a footnote regarding another "Wantzell"; but nowhere does Poggendorff refer to a "Wantzel." Text-books on algebra and the theory of equations do not give Wantzel's full name. The reader is thus left without positive information as to the author of "Wantzel's proof." His name suggests German nationality, as does the name of "Mannheim," of slide-rule fame. Yet both these men were born in Paris and passed their lives at the Polytechnic School there.* Born in 1814, Wantzel died prematurely in 1848. He is the "Pierre Laurent Wantzell" of Poggendorff but in his published articles his name is always spelled

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* On the life of Wantzel, see Barré de Saint-Venant in *Nouvelles Annales de Mathématiques* (Terquem et Gerono), vol. 7 (1848), pp. 321–331; A. de Lapparent in Ecole polytechnique, Livre du Centenaire, 1794–1894, vol. I., Paris, 1895, pp. 133–135, see also pp. 63–65, 190; Gaston Pinet's Ecrivains et Penseurs Polytechniciens, 2e éd., Paris, 1902, p. 20; Charles Sturm in *Comptes rendus hebdomadaires des Séances de l'Académie des Sciences*, Paris, vol. 28 (1849), pp. 66, 67.

"Wantzel." Pinet says of him: "Endowed with extreme vivacity of impressions and with truly universal aptitudes, he carried off the prize for a French dissertation and a Latin dissertation at a general competition and, the year following (1832), entered with first rank the Polytechnic School—a double success before unheard of. He studied with inveterate zeal German and Scotch philosophers; he threw himself into mathematics, philosophy, history, music, and into controversy, exhibiting everywhere equal superiority of mind." He became élève-ingénieur des Ponts et Chaussées, then ingénieur; he was appointed répétiteur about the time when such positions were held by Comte, Transon, Bertrand, Bonnet, Catalan, Leverrier, and Delaunay. In 1843 he became examinateur d'admission. Saint-Venant says of him: "He was blameworthy for having been too rebellious to the counsels of prudence and of friendship. Ordinarily he worked evenings, not lying down until late; then he read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours until he was married. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death."

The Royal Society Catalogue of Scientific Papers quotes the titles of 18 papers by Wantzel, and of three more which he brought out jointly with Saint-Venant.

General Quintic Insolvable by Radicals.

As previously stated, Wantzel's most noted scientific achievement is found in his paper "De l'impossibilité de résoudre toutes les équations algébriques avec des radicaux" in the *Nouvelles Annales de Mathématiques*, volume 4 (1845), pages 57–65. The second part of this proof, which involves substitution theory, is reproduced in Serret's Algèbre supérieure.*

At the beginning of his article, Wantzel expresses himself on the proofs of Abel and Ruffini as follows: "Although his (Abel's) demonstration is at bottom exact, it is presented in a form too complicated and so vague that it is not generally accepted. Many years previous, Ruffini, an Italian geometer, had treated the same question in a manner much vaguer still and with insufficient developments, although he had returned to this subject many times. In meditating on the researches of these two geometers and with the aid of principles which we established before,[*] we have arrived at a form of proof which appears so strict as to remove all doubt on this important part of the theory of equations." His previous paper, to which reference is made, deals with algebraic incommensurables. Wantzel says that the main theorem set forth therein had been previously established by Abel and again by Liouville. Thus, Wantzel expresses special indebtedness to Abel and Liouville, while to Ruffini he makes acknowledgment only in a general way as indicated in the passage quoted above. A different estimate of indebtedness has been made more recently by H. Burkhardt[†] and E. Bortolotti.[‡] They claim that the proof which goes by the name of Wantzel is essentially Ruffini's proof of 1813. Inasmuch as Wantzel's proof has been generally accepted as altogether valid, the implication is that Ruffini's proof is equally valid.

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* In the fifth edition, 1885, the part proof occurs in vol. II, pp. 512–517.

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[*] *Nouvelles Annales* (Terquem), vol. 2 (1843), pp. 117–127.

[†] H. Burkhardt, "Die Anfange der Gruppentheorie und Paolo Ruffini," *Zeitsch. f. Mathematik u. Physik*, 37. Jahrg., Supplement, Leipzig, 1892, pp. 119–159. Burkhardt says (p. 156): "Es braucht wohl kaum noch ausdrücklich hervorgehoben zu werden, dass diese Fassung des Unauflosbarkeitsbeweises sich in allen wesentlichen Punkten mit derjenigen deckt, welche als 'Wantzel'sche Modification des Abel'schen Beweises' in den Lehrbuchern mitgeteilt zu werden pflegt." Again he says (p. 159), "diesen Beweis hat er (Ruffini) nicht nur zuerst durchgefuhrt, sondern ihn auch nach verschiedenen Umarbeitungen auf die einfache Form gebracht, welche Wantzel zugeschrieben zu werden pflegt."

[‡] In the Carteggio di Paolo Ruffini con alcuni scienziati del suo tempo, Roma, 1906, p. 15 (303), Bortolotti says in a footnote that Ruffini's Riflessioni intorno alla soluzione delle equazioni algebraiche generali, Modena, 1813, "contiene la dimostrazione, che per lungo tempo attribuita al Wantzel, fu sempre considerata come la più facile e la più convincente del teorema di Ruffini." Bortolotti expresses himself more fully in his discourse, Influenza dell' Opera Matematica di Paolo Ruffini (Modena), 1902, p. 42: "E sarebbe anche strano che egli (Abel) accusasse la dimostrazione di Ruffini di esser troppo complicata; laddove essa è tanto semplice che, quando il *Wantzel* volle ridurre il teorema a forma facile e piana, fu costretto a riprodurre, nelle sue linee generali, la redazione del Ruffini. L' ingiustizia delle umane cose ha però voluto, e vuole tuttora, che la dimostrazione che il Ruffini ha data del teorema da lui stesso scoperto, sia stata battezzata col nome di *Dimostrazione di Wantzel del Teorema di Abel*. Quella disgraziata denominazione, introdotta dal Serret nel suo rinomato libro di *Algèbre supérieure* è stata ricopiata di testo in testo fino ai nostri giorni: e si trova anche nell' ultimo e più reputato testo di Analisi, quello del Picard, uscito nel 1896 quattro anni dopo la piena ed irrefutabile dimostrazione data dal Burkhardt della priorità del Ruffini."

The question arises, is Wantzel's proof really the same as Ruffini's of 1813? Before answering this query, the question must be considered, how much must be included as constituting an integral part of the proof? Is only the portion involving substitution theory to be taken into account, or is the introductory part relating to irrational algebraic functions and the general expression for the root form of an algebraically solvable equation to be considered as well? Burkhardt and Bortolotti seemingly look upon the application of substitution theory as the only essential part of the demonstration. But, as Netto says, the "proof of the impossibility of an algebraic solution of general equations above the fourth degree can never be obtained from the theory of substitutions alone." [*] Moreover the state of algebra at the beginning of the nineteenth century required that both parts be investigated, to establish the impossibility. With Ruffini and Abel, and also with Wantzel, the establishment of the general root form of algebraically solvable equations was the most difficult and serious part which occupied the larger portion of the space devoted to the proofs. Both parts received consideration by William Rowan Hamilton,[†] L. Kronecker,[‡] O. Hölder,[§] and J. Pierpont.[||] Sylow[¶] and Hölder point out that there is an unproved assumption in the *first part* of Ruffini's proof. Neither Ruffini nor Abel were ever seriously criticized for lack of rigor in the strictly substitution theory part, but both were criticized on the other parts. Abel committed an error in the statement of the theorem relating to an algebraic function of the μth order and mth degree, which was corrected by L. Königsberger in 1869.[**] This error had disturbed William Rowan Hamilton, who, in his long article[††] "On the argument of Abel," declared that the error "renders it difficult to judge of the validity of his (Abel's) subsequent reasoning." However, J. Pierpont[‡‡] rightly remarks that this slip "does not affect in the least the validity of the proof." Abel establishes the fundamental theorem, first enunciated by Vandermonde: "If an equation is algebraically solvable, then one can always give its root such a form that all algebraic functions of which it is composed, can be expressed by rational functions of the roots of the given equation." This theorem is fundamental also in Ruffini's research but he failed to give it convincing proof. It is to Abel's credit to have given it a binding demonstration. Abel gives a classification of algebraic functions involving radicals. Wantzel follows Abel, but gives a shorter treatment containing only what is needed for his immediate purpose. In his paper of 1843, Wantzel deals with the properties and classification of radicals and of rational functions of radicals that involve root extractions of real but not of imaginary "quantities." In his paper of 1845, in which the subject is continued and the Ruffini-Abel theorem is established, he states that the results previously proved for radicaux numériques apply equally to radicaux algébriques and proceeds on this assumption. Wantzel's treatment of this first part of his proof is not always happy; Serret in his Algèbre supérieure prefers to adhere to Abel's exposition. There is nothing in Ruffini that directly suggested to Wantzel his mode of classifying algebraic functions. On the other hand, Wantzel's indebtedness to Abel is strongly evident. Nor does Wantzel attempt to conceal this fact. Besides the general statement of indebtedness to which we referred earlier, Wantzel makes special reference to Abel in two places: one where he says that his names for the different classes of radicals are pretty nearly the same as Abel's; the other where he states that his fundamental theorem on radicals was proved before him by Abel and Liouville.*

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[*] E. Netto, Theory of Substitutions, translated by F. N. Cole, Ann Arbor, 1892, p. 240.

[†] W. R. Hamilton, "On the argument of Abel," *Transactions of the Royal Irish Society of the year 1839*, p. 248.

[‡] L. Kronecker, *Monatsbericht d. k. p. Akademie d. Wiss. zu Berlin*, 1879, p. 205.

[§] Encyklopädie d. math. Wissensch., 1. Band, p. 504.

[||] J. Pierpont in *Monatshefte f. Math. u. Physik*, VI (1895), pp. 37–51.

[¶] Oeuvres of Abel, edition by Sylow and Lie, II, 1881, p. 293.

[**] L. Konigsberger, *Math. Annalen*, vol. 1 (1869), pp. 168, 169.

[††] W. R. Hamilton, loc. cit., p. 248.

[‡‡] J. Pierpont, loc. cit., p. 47.

The Ruffini publication of 1813, previously referred to, consists of viii + 140 pages. The impossibility of an algebraic solution of the general equation of a degree higher than the fourth is given in Part I; Part II treats of the impossibility of solution by the aid of certain transcendental expressions. Here again, a comparatively small portion concerns itself with the part of the proof involving substitution theory. The larger part of the book is devoted to the study of the properties of algebraic functions and of a certain class of transcendental functions (espressione trascendente esatta) which; among others, includes trigonometric and logarithmic functions, but is not sharply defined; nor does the many-valuedness of these transcendental functions receive adequate attention. The proof of the impossibility of solution by an espressione trascendente esatta possesses therefore little value.

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* See Wantzel in *Nouvelles Annales de Mathématiques*, Tome II, Paris, 1843, pp. 125, 127.

If, for the moment, we leave out of consideration everything that Ruffini says about transcendental functions, and confine ourselves to his proof of the impossibility of an *algebraic* solution, we find that, although inconclusive in some important details, it must be admitted to his everlasting credit that the general outline of his demonstration is correct. The part of the proof regarding the form which an algebraic root must take (if such an algebraic root exist) is the weak part. Certain theorems necessary for his proof are not established, or rest upon illusory arguments. Letting P be a rational function of the coefficients of the given equation, he introduces the first radical Q by the relation $Q^p = P$ (p prime), then he lets F_1 be a rational function of P and Q, and defines the second radical R by the relation $R^q = F_1$ (q prime), etc. His general root form is $x_n = F\ (P, Q, R, S, \text{etc.})$, where (page 12) F is a rational function of $P, Q, R, S, \cdots$. It is *assumed* by Ruffini that if the equation can be solved by algebra, the form of its roots must be the one given here.

The part of Ruffini's proof involving substitution theory is free of fault and is in outline as follows:

(1) Let P be a symmetric function of the roots $x_1, x_2, \cdots, x_5$ and unaltered by the cyclic substitution $s = (12345)$. Applying $s, s^2, \cdots$ to y_1, where $y_1{}^p = P$, gives $y_1{}^p = y_2{}^p = \cdots = y_5{}^p$, and $y_2 = \beta y_1, \cdots, y_5 = \beta^4 y_1, y_1 = \beta^5 y_1$; hence $\beta^5 = 1$.

(2) P is unaltered by $(123) = \sigma$. Applying $\sigma, \sigma^2, \cdots$ to y_1, we get by the above process $y_a = \gamma y_1, y_{a+1} = \gamma^2 y_1, y_1 = \gamma^3 y_1$ and $\gamma^3 = 1$,

(3) The substitutions $(12345)(123) = \tau, \tau^2, \cdots$ applied to y_1 give similarly $\beta^5\gamma^5 = 1$. Hence $\gamma = 1$, and $y_1 = y_a = y_{a+1}$.

(4) The substitutions $(345) = \phi, \phi^2, \cdots$ applied to y_1 give similarly $y_c = \delta y_1, y_{c+1} = \delta^2 y_1, y_1 = \delta^3 y_1$. Hence $\delta^3 = 1$.

(5) The substitutions $(12345)(345) = \pi, \pi^2, \cdots$ applied to y_1 yield $\beta^5\delta^5 = 1$, hence $\delta = 1$ and $y_1 = y_c = y_{c+1}$. Hence $y_1 = y_a = y_c$. Applying σ to these yields $y_1 = y_a = y_2$, hence $\beta = 1$ and Q is unaltered by $s, s^2, \cdots$. The same conclusion can be reached for the algebraic irrationals $R, S, \cdots$, and

finally for F. Hence $x_n = F(P, Q, R, \cdots)$ is absurd, since its right member is unaltered by certain substitutions while the left member is altered.

The part of Wantzel's proof involving substitution theory is as follows:

(1) Let p be a symmetric function of the roots $x_1, x_2, \cdots x_5$, and y a rational function of the roots, and $y^n = p$. The transposition (12) yields $y_1 = \alpha y$, $y = \alpha y_1$; hence $\alpha^2 = 1$ and $n = 2$; the first radical appearing in the root form is a square root.

(2) In $z^n = p_1$ let z be a rational function of the roots which is altered by a cyclic substitution of three letters, but p_1 is unaltered by it. We obtain $z_1 = \alpha z$, $z_2 = \alpha z_1$, $z = \alpha z_2$; hence $\alpha^3 = 1$ and $n = 3$.

(3) Applying to this same z a cyclic substitution of five letters, he proves that $\alpha^5 = 1$. But $\alpha^3 = 1$; hence $\alpha = 1$.

(4) Hence when the degree of the equation exceeds four, z is invariant under a cyclic substitution of three letters. The relation between the roots $x_1 = \psi(x_1, x_2, x_3, x_4, x_5)$ must be an identical equation. This is impossible since the left member is altered by (123) and the right member is not.

The substitution theory as applied by Wantzel is reminiscent of that of Ruffini, yet is far from identical with it. Wantzel stands no closer to Ruffini in this second part of the proof than he stands to Abel in the first part. It is what one might have expected Wantzel to contribute, after "meditating on the researches of the two geometers." The proofs of Ruffini and Wantzel differ altogether in the first part. Hence the claim put forth by Burkhardt and Bortolotti, that Wantzel's proof is the same as Ruffini's, is wholly unsupported by the facts as regards the first part and is too sweeping as regards the second part.

The "Irreducible Case" in Cubics.

Quite forgotten are the proofs given by Wantzel of three other theorems of note, viz., the impossibility of trisecting angles, of duplicating cubes, and of avoiding the "irreducible case" in the algebraic solution of irreducible cubics. For these theorems Wantzel appears to have been the first to advance rigorous proofs. To be sure, Paolo Ruffini, in his booklet of 1813 (pages 54–57), had put forth a proof that the "irreducible case" is unavoidable, but in the absence of satisfactory demonstration that the root must take a certain form, it is open to objection. Wantzel proves this theorem at the end of his paper of 1843, previously referred to. He writes a root of the irreducible cubic in the form of an entire radical function of the nth kind (espèce), $x = A + Bu + \cdots + Mu^{m-1}$, where $u = \sqrt[m]{a}$, a is a radical function of the $(n - 1)$th kind, while $A, B, \cdots, M$ may be such functions of the nth kind but are of a degree inferior to x. Roots of imaginary numbers do not occur in x. He shows that B may be taken ± 1. Substituting for u in x the roots of the irreducible equation $u^m - a = 0$, he finds that $m = 3$ or 2. If $m = 3$ and $\alpha^3 = 1$, he proves that

$$x + \alpha x_1 + \alpha^2 x_2 = \pm 3u.$$

Since, by supposition, x, x_1, x_2, u are all real, it follows that $x_1 = x_2$, which is impossible since the given cubic is irreducible. He shows that $m = 2$ is likewise impossible. Hence the "irreducible case" cannot be avoided. Nearly half a century later proofs were given of this theorem by Mollame (1890), Hölder (1891), and Kneser (1892).

Duplication of Cubes and Trisection of Angles.

These problems are taken up in Wantzel's article in *Liouville's Journal*, volume 2 (1837), pages 366–372, on the "means of ascertaining whether a geometric problem can be solved with ruler and compasses." He shows first that problems solvable by ruler and compasses can be solved algebraically by a series of quadratic equations, (*S*) $x_i^2 + A_{i-1}x_i + B_{i-1} = 0$ $(i = 1, 2, \cdots, n)$ where A_0 and B_0 are rational functions of given numbers $p, q, \cdots$, and where A_i and B_i are rational functions of $x_i, x_{i-1}, \cdots, x_1, p, q, \cdots$. Secondly, he shows that, if in A_{n-1} and B_{n-1} we substitute in succession the two values of x_{n-1} obtained by solving $x_{n-1}^2 + A_{n-2}x_{n-1} + B_{n-2} = 0$, and if thereupon we multiply together the two resulting expressions for the left member of $x_n^2 + A_{n-1}x_n + B_{n-1} = 0$, we obtain an equation of the fourth degree in x_n. Its coefficients are rational functions of $x_{n-2}, x_{n-3}, \cdots, x_1, p, q, \cdots$. Repeating this process by eliminating successively $x_{n-2}, \cdots, x_1$, he obtains one equation of the degree 2^n in x_n, the coefficients of which are rational functions of $p, q, \cdots$. Thirdly, he shows that an equation of the degree 2^n, resulting from the least possible number of quadratic equations necessary to solve a given problem by ruler and compasses, is irreducible. Fourthly, he considers tests for determining whether a proposed irreducible equation of the degree 2^n can be solved by a series of square root extractions. This is done by equating the coefficients of the proposed equation with those of the general equation of the degree 2^n obtained from the system (S). If thereby the coefficients of the quadratic equations (S) can be obtained by the extraction of square roots, but of no higher roots, then the proposed equation can be solved in that manner.

Wantzel then remarks that the equation $x^3 - 2a^3 = 0$, arising in the problem of the duplication of a cube, is irreducible, but is not of the degree 2^n; hence the cube cannot be doubled in volume by a construction with ruler and compasses. He draws the same conclusion for the equation $x^3 - \frac{3}{4}x + \frac{1}{4}a = 0$, on which depends the trisection of an angle. His own words are: "Cette équation est irréductible si elle n'a pas de racine qui soit une fonction rationnelle de a et c'est ce qui arrive tant que a reste algébrique; ainsi le problème ne peut être résolu en général avec la règle et le compas. Il nous semble qu'il n'avait pas encore été démontré rigoureusement que ces problèmes, si célèbres chez les anciens, ne fussent pas susceptibles d'une solution par les constructions géométriques auxquelles ils s'attachaient particulièrement."

Saint-Venant admits Wantzel's claim of priority and adds that somewhat later Charles Sturm simplified the proofs but did not publish them. So far as now known, Wantzel's priority in publishing detailed, explicit and full proofs of the impossibility of doubling cubes, of trisecting angles and of avoiding the "irreducible case" in the cubic is incontested.

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ADDITIVE FUNCTIONS OF A POINT SET.

Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire. Par C. DE LA VALLÉE POUSSIN. Paris, Gauthier-Villars, 1916. viii + 154 pp.

THE general notion of an additive function of a point set is one of the most important introduced by the present French school of mathematicians. It is due to H. Lebesgue. An additive function of a point set is one whose value on a sum of sets is the sum of its values on each term. These terms, the number of which may be infinite, have in pairs no common element. The volume under review (one of the Borel monographs) is devoted to the theory of additive functions of a point set; it contains the matter of lectures delivered at the College of France between December, 1915, and March, 1916. The author had already treated the problem in his Harvard lectures, which were partially published in 1915 in the *Transactions of the American Mathematical Society.* Some of the results had also appeared in volume II of his Cours d'Analyse, third edition. In the present volume the treatment is rendered distinctly more complete and satisfying through the use of new methods and the derivation of new results.

In these lectures we have a careful analysis of the notion of additivity and the derivation of the consequences, singularly precise and interesting, which follow from the sole property of a function implied in this notion.

The simplest and earliest known additive function of a point set is its measure. The definition was given by Borel in 1898. It furnishes the point of departure for the entire theory of additive functions. It is this function which is considered in the beginning of the first of the three (nearly equal) parts of the present monograph. (Lebesgue integrals are treated in the latter portion of this part.) The measure is a non-negative function whose value is given on certain point sets called elementary figures and is then to be defined on other sets in such way as to satisfy the requirement of being additive. The point sets on which this requirement is satisfied are then the measurable sets.

If a function possesses the property of being additive only for a finite number of terms it is said to be additive in a restricted sense; if for an enumerable infinitude of terms it is said to be additive in a complete sense. The essential progress obtained by Borel and Lebesgue in the theory of measure consists in their having realized additivity in the complete sense. The first idea of this theory is due to Borel. The proper work of Lebesgue commences only with definite integrals.

The question of measure from the point of view of Borel's contribution gives rise to the following difficult problem: "a function being given on certain particular point sets, such as the elementary domains, does there exist an additive function which coincides with the preceding on these domains?" It is this fundamental question which is resolved in the second part of these lectures. One reaches the conclusion that there is complete equivalence, under the condition of continuity, between an additive function of a point set and a function $f(x)$ of bounded variation.

A function $g(e)$ of a point set e is said to be continuous if its value approaches zero with the diameter of e, and to be absolutely continuous if its value approaches zero with the measure of e. Among the additive functions of a point set the most important are those which are absolutely continuous. The indefinite integrals of Lebesgue possess these two properties of additivity and absolute continuity and are characterized by them. The derivatives of absolutely continuous functions of a point set are studied (also in the second part) by the new method of networks (réseaux) already utilized in the Harvard lectures. We return later to a discussion of this method.

The questions mentioned in the foregoing paragraphs are throughout of a metric nature. Questions of a more exclusively descriptive kind, but intimately related to the foregoing, are treated in the third part. Here the author takes up the separation of functions into the successive classes of Baire, simplifying and completing the exposition of this interesting theory by the introduction of new methods and the addition of new results. The relation of Baire's classification to the other matters treated in the book is brought out by showing the identity of the class of functions measurable in the sense of Borel and the class of functions contained in Baire's classes of all orders.

Thus the questions treated in this work belong to the recent theory of functions of which the founders are Borel, Baire and Lebesgue. In the course of his profound researches Baire has set off a real functional domain which appears to suffice for all the needs of analysis and seems to be such that every generalization is condemned beforehand to be vain and sterile. The functions of this domain possess precise common properties. The general methods of analysis are applicable to them; and their theory, already rich in results, may be considered as the *general theory of functions of a real variable.*

Thus we have here a fundamental advance from the general point of view of the philosophy of mathematics. In this progress the work of Lebesgue has been central. More than any other, Lebesgue has contributed to put in evidence the unity of this theory of functions of real variables and to beautify it by bringing out that aesthetic character which delights the artistic spirit of the pure mathematician. The author has achieved his aim in the book if the reader recognizes this character in his pages.

Such are the general features of this monograph in which there is much to please one and nothing to annoy. It remains to give a brief account of the novelties in the author's methods and results.

In the first part, which deals with measurable sets and the integrals of Lebesgue, the extent of novelty is not so great as in the other two parts. Here the author treats general notions about point sets, measure of sets and measurable functions, and Lebesgue integrals, devoting a chapter to each of the three topics. On page 10 is given a generalization of the usual definitions of interior, exterior, and frontier points of a set (see also page 105). But the chief novelty to which attention may be called is in the notion of the characteristic function of a point set, a notion already utilized by de la Vallée Poussin in his *Transactions* memoir.

Let E be a point set in a space $x, y, \cdots$ of one or more dimensions and let CE denote the set complementary to E in such a space. Then the characteristic function or the characteristic of the point set E is the function $\varphi(x, y, \cdots)$ which has the value unity at each point of E and the value zero at each point of CE. It is clear that $1 - \varphi$ is the characteristic of CE. If $\varphi_1, \varphi_2, \cdots$ are the characteristics in order of $E_1, E_2, \cdots$, these sets having in pairs no common elements, then

$\varphi_1 + \varphi_2 + \cdots$ is the characteristic of $E_1 + E_2 + \cdots$. Also, $\varphi_1\varphi_2\cdots$ is the characteristic of $E_1E_2\cdots$ without condition on $E_1, E_2, \cdots$. To find the characteristic of the sum $E_1 + E_2 + \cdots$, when no condition is put on the component sets, observe that this sum is the complement of the set $CE_1 \cdot CE_2 \cdot CE_3 \cdots$; hence its characteristic is

$$1 - (1 - \varphi_1)(1 - \varphi_2)(1 - \varphi_3)\cdots.$$

Through the use of characteristic functions the problem of infinite sums and products of point sets is readily reduced to that of the passage to a limit so that certain fundamental properties of such sums and products may at once be read off by means of corresponding properties of limits.

It is not difficult to show (see page 36) that the characteristic function of a point set which is measurable in the sense of Borel belongs to one of Baire's classes. This leads to a natural classification of point sets which are measurable in the sense of Borel, the class number of the set being the same as that of the corresponding characteristic function in the classification of Baire. Certain relations among the classes of point sets then follow at once from corresponding relations among the classes of Baire.

In approaching the theory of measure in the second chapter, de la Vallée Poussin develops first of all the theory of measure for enumerable sums of intervals and closed sets of points and then through the aid of this theory defines the measures of more general sets. In this connection it is of interest to note that Bliss (this BULLETIN, volume 24 (1917), pages 1–47) has returned to the more direct methods of Borel and Lebesgue which found the theory of measure entirely upon the measurability of enumerable sets of intervals. Through aid of the improved methods of de la Vallée Poussin, Bliss is able to approach the subject in a way which is especially concise and clear. At the same time he establishes without additional complication the foundations of the theory of the positive additive functions of a point set, of which the measure function is a special case. The reader of de la Vallée Poussin will find it useful to compare the improved exposition of Bliss.

The second part of the monograph deals in general with additive functions of a point set, its three chapters having in order the following titles: general notions concerning derivatives and networks; absolutely continuous and additive functions of a point set, indefinite integrals; additive functions of a normal set. This part is rich in novelty. One meets here the term symmetric derivatives (page 59), the particularly fruitful notion of networks (pages 61 ff.) and of derivatives on a network, and again (page 74) the major and minor functions of a summable function $f(x)$ which de la Vallée Poussin first introduced in 1909 in the second edition of volume I of his Cours d'Analyse. The greater part of Chapter VI is based on the author's *Transactions* memoir; the exposition is simplified and the results are frequently generalized or rendered more precise.

The new method of networks already employed by de la Vallée Poussin in his Harvard lectures is now rendered much more useful through the notion of conjugate networks. By aid of this, certain demonstrations are made in a simpler and more natural manner than before and the form of the exposition is improved in many respects. The novelty associated with this method constitutes one of the two central features of the monograph and is perhaps the most interesting illustration in it of the author's insight into the problems with which he is dealing. Fortunately a clear, excellent and readable connected exposition of these new ideas is given on pages 61–66, so that it is unnecessary for us to attempt a summary here in our limited space. The reader who wishes to follow up some of the main uses of the method elsewhere in the monograph may consult pages 67–82 and 96–104.

The second central feature of the monograph is to be found in the third part in the treatment of Baire's classification of functions. In his thesis (1899) Baire demonstrated the following theorem: A necessary and sufficient condition that a discontinuous function f shall be of class 1 on a bounded and perfect set P is that it shall be pointwise discontinuous on every perfect set Q contained in P. This is called Baire's theorem. It is intimately associated with the problem of Baire, namely: Given a function f of class 1, to construct a sequence of continuous functions $f_1, f_2, \cdots$ such that f_n approaches f as n approaches infinity.

Baire proved the last preceding theorem by the direct process of resolving the corresponding problem. The first demonstration was confined to a function of a single variable. Lebesgue (1899) proved the theorem for a function of several variables by a process of reduction to the case of a single variable. Baire extended his original method so as to render it applicable to functions of several variables. Finally he gave a systematic exposition of his theory in 1905 in his Leçons sur les Fonctions discontinues.

The methods of Baire make appeal to the transfinite. Lebesgue* has given demonstrations of Baire's theorem not requiring the intervention of the transfinite. He states the condition in the form: A necessary and sufficient condition that a discontinuous function f shall be of class 1 on a bounded and perfect set P is that the set of points where $f > A$ and the set where $f < A$ shall be sums of closed sets for every A. The methods of Lebesgue do not afford a solution of Baire's problem.

In the present monograph we have a new demonstration of Baire's theorem and a new and simple resolution of Baire's problem. The method of exposition is a sort of synthesis of those employed by Lebesgue and Baire. It completes those of Lebesgue and, by the introduction of an auxiliary theorem, notably simplifies that of Baire.

In the theorems of Baire and Lebesgue we have two very different expressions of the condition that a function shall be of class 1. If one seeks to generalize these theorems and to obtain analogous distinctive characteristics of functions of general class α ($\alpha > 1$), he finds that Baire's form generalizes only imperfectly (see page 144) while an adequate and complete generalization of Lebesgue's form is available (see page 141). Lebesgue's method of treatment is here followed in the main, with improvements in certain parts and the introduction of some novel results.

For the statement of the generalization one needs to employ the notion of sets O and F of Lebesgue with respect to a bounded and perfect set P. De la Vallée Poussin puts these definitions in the following form: *A set E is an F of class α, if there exists a function θ, defined on P and of class $\leqq \alpha$, such that the set E coincides with the set of points of P on which $\theta = 0$; a set E is an O of class α, if there exists a function θ, defined on P and of class $\leqq \alpha$, such that the set E coincides with the set of points of P on which $\theta \neq 0$.*

The generalized condition of Lebesgue may now be stated in the following form: A necessary and sufficient condition that a function f (finite or not) shall be of class $\alpha > 0$ on P is that for every constant a the set where $f > a$ and that where $f < a$ shall be sets O of class α and shall not be both of class less than α; or, what amounts to the same thing, that the (complementary) set where $f \leqq a$ and that where $f \geqq a$ shall be sets F of class α and not both of class less than α.

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* Borel's Leçons sur les Fonctions de Variables réelles et leur Représentation par les Séries de Polynomes, Note II; *Bulletin de la Société mathématique de France*, 1905.

The monograph closes with a demonstration (at bottom agreeing with that of Lebesgue but expressed in somewhat more elementary form) of the theorem which asserts the existence of functions of all of Baire's classes.

In view of the numerous definitions of integration which have been given and the present state of unrest in this range of ideas, one desires to consider Lebesgue integrals not merely in themselves but also in their relation to other types of integrals. Such a comparative treatment does not fall within the scope of de la Vallée Poussin's monograph. Fortunately a careful analysis of this sort is now (readily) accessible in Hildebrandt's recent paper (this BULLETIN, volume 24 (1917–1918), pages 113–144, 177–202).

In the monograph under review and in the two articles by Bliss and by Hildebrandt already referred to we have recent treatments of the general problem of integration which supplement and complete each other in a useful way. De la Vallée Poussin develops the theory of Lebesgue integrals in intimate association with several fundamental matters to which at bottom they are closely related, so that through this monograph one may obtain a view of this domain of mathematics in its proper relation to other fields and in appropriate perspective as to its own parts among themselves. Bliss gives a particularly compact, but at the same time clear, development of the theory of Lebesgue integrals and the intimately associated theory of measure, the march of ideas in his treatment being singularly direct and leading the reader to the goal without loss of energy due to indirect processes and hence with a minimum of effort. Hildebrandt's purpose is to discuss briefly the several definitions of integration and particularly to consider their relations one to another. He gives careful attention to the distinctive features of each definition, to the question of equivalence, and to the problem of further generalization.

Equivalences among integrals are of two types. The first may be called complete equivalence; it is one in which the two symbols of integration operate upon the same class of functions and give the same integral value for the same function. The other is in reality a pseudo-equivalence expressed in the fact that an integral of one kind may be transformed into an integral of another kind, the functions integrated in the two cases being different. A careful and instructive analysis of these equivalences is given by Hildebrandt.

To one result of this analysis it is desirable to have attention sharply directed. The Stieltjes integral seems destined to play in the future a rôle of central importance in processes of integration and summation. The Lebesgue integral when introduced received almost immediate attention and recognition and found its way rapidly into the main current of mathematical thought; but the Stieltjes integral has been singularly neglected notwithstanding its inherent simplicity and naturalness. Through the summary of its properties given by Hildebrandt and the applications mentioned one is convinced of its central importance and is led to expect it to assume a new place in mathematical thought. In this connection it is of particular interest to note also Hildebrandt's extension of the Stieltjes integral modelled on the Lebesgue extension of the Riemann integral.

We conclude with the following list of misprints in de la Vallée Poussin's monograph: page 20, last line, write $m(F_1 + F_2) = mF_1 + mF_2$; page 28, second theorem, write $(f \geqq A)$ instead of $(f \geqq a)$; page 34, line 16, write $\omega 2$ instead of ω_2; page 96, end of second paragraph, write Df instead of DF; page 126, line 9 below, write "Une" instead of "Uue"; page 133, line 2, write "de classe $\leqq \alpha$."

R. D. CARMICHAEL.

SHORTER NOTICES.

Leçons de Mathématiques Générales. Par L. ZORETTI. Paris, Gauthier-Villars, 1914. 8vo. xvi + 753 pp.

Exercices numériques de Mathématiques. Par L. ZORETTI. Paris, Gauthier-Villars, 1914. 8vo. xv + 125 pp.

THE author of this text and its accompanying set of exercises is well qualified for the task. Formerly instructor in the lycée, contributor to the theory of functions by several important memoirs, professor of mechanics, he is fully equipped to add to several already existing treatises on mathematics in general another of the particularly successful French treatises of this kind. His special object was to write a book that would furnish the necessary knowledge for the most widely different classes of students, whether they intended to enter experimental science, engineering, the ranks of teaching, or some of the professions. His goal is that of applications, and everything is included that might be useful for practical mathematics, everything is excluded that cannot at once justify its applicability elsewhere.

This program, to be realized in a manageable space, requires a presentation which is as intuitive as possible, but still attacking with all frankness demonstrations which must be given, and further considering carefully the conditions under which the results that are arrived at may be applied, and the significance of these results. Exactness in definition is also necessary in order that the student may clearly understand what he is doing. The student is thrown upon his own resources, particularly in the exercises, many of which demand careful construction and measurement by the student himself. We might say that a certain reasonable amount of laboratory mathematics is insisted upon in order that the student may learn not only to apply mathematics to data given in some way, but to derive the data to which the mathematics is to be applied.

The first 157 pages are devoted to a survey of analytical geometry, including the theory of vectors. This grouping of the subjects is perfectly proper inasmuch as analytical geometry is merely a rather primitively developed vector analysis. The first chapter defines vector (directed segment), the equivalence of vectors which are on the same axis, and the equipollence of vectors which are on parallel axes. Couple is included, as well as opposed vectors. Addition and subtraction of vectors on the same axis is defined, as well as the same processes for vectors whose axes are concurrent. These notions are extended to directed arcs and angles. The theorems of projection follow naturally.

The next chapter defines and illustrates the different kinds of coordinates for surfaces and for space. The method of the book is very well shown in the way in which the systems of curvilinear coordinates are defined and their uses indicated. This kind of coordinate is seldom mentioned in the beginning of the usual texts. The elementary formulas for a line are developed, and the determination of mean distances for n points is introduced as an example, thus putting the student in touch at once with possible uses for his knowledge. The coordinates of a vector are considered next, followed by the determination of any figure, bringing in at the same time Euler's angles, and degrees of freedom. The determination of lines on surfaces, and surfaces themselves follow. Then the methods of changing the system of coordinates are developed. These ideas are then to be applied to such exercises as the following: Choose proper parameters for the following figures: segment of a line located in a given plane; circle in space; given cone of revolution, the vertex in a given plane; circle tangent to a given line in space; coordinates of a point on a given line or surface.

It seems to the reviewer that such a method as this marks a distinct advance over the usual one of explaining the rectangular and polar coordinates followed by a few perfunctory plottings, and then a development of formulas for the line and plane. The student is generally supposed to seize instantly the idea of coordinates and their significance, and to be able to reason by their means with no further study of them. Their significance as parameters or numbers which may arise in an infinity of ways, different sets being useful for different purposes, is seldom pointed out, and the student is generally unable to state his problem in any other way than by the conventional methods of x, y, z or r, θ, φ. When he comes to integration in space he is usually helpless.

Chapter three develops the fundamental formulas of the line and plane. Chapter four considers the theory of vectors, moments, resultants, resultant moments, equivalent systems, invariants, central axis, couples, coplanar vectors, and parallel vectors. The student is early equipped with useful knowledge for statics. Two chapters follow on circles and conics. In the latter the approximate construction by circular arcs is described. In the manual of exercises the second series contains other methods of drawing these curves, with some original theorems on them easily deduced from the methods. Chapters seven and eight consider quadrics and some of the common curves and surfaces.

The second part of the work is given up to algebra, theory of functions, and derivatives. It occupies 342 pages, the greater part being concerned with functions. The algebraic topics cover complex numbers and their representation, the binomial theorem, determinants, and elimination. A chapter is devoted to infinite series, with the convergence tests of D'Alembert, Cauchy, and the rule that is derived by comparison of the series with the series Σn^{-p} where $1 < p$. These are characterized for practical purposes as follows: The last reveals only slowly convergent series, the second is not easy of application, so that the rule of D'Alembert is about the only practical one.

The notion of function is based upon dependence, but the illustrations are of natural dependence and not of the artificial type often seen in such examples. The author then defines, and considers the uses of, entire functions, simply periodic functions (a vastly better name than harmonic functions), the exponential function, circular functions as exponential, hyperbolic functions, logarithmic function, inverse circular functions, inverse hyperbolic functions, with some closing remarks on graphs when the quantities represented by x and y are not of the same kind.

In discussing differentiation examples are freely used to develop some very fundamental notions, often not mentioned in the beginning works on calculus studied by engineers and other practical students. For instance, it is shown by a wave curve that lies in a very narrow strip, that while functions which have derivatives which lie close together do not differ much themselves, on the other hand functions which lie close together may have derivatives very far apart. For this reason integration never has the hard conditions to meet which restrict derivation. Another instance of departure in a useful way from other elementary presentations is in the definition and use of the directed derivative of scalar and vector point functions, notions much more useful than a large set of complicated algebraic formulas for differentiation. A chapter of variations of functions presents several well known curves and some of their peculiarities. A short chapter on development in series contains the essentials. Derivatives are applied to curves, surfaces, and movements, including relative motion. A chapter is given to theory of equations and their solutions, and a chapter to numerical and graphical calculation.

The third part of the work covers 248 pages, and considers the integral calculus and its applications. The integral is defined as the limit of a sum, making the definite integral properly the foundation of the whole subject. The connection with the primitive of the simple differential equation $dy = f(x)dx$ is immediately brought out, and some of the formulas of differentiation, as $d\cdot uv = vdu + udv$, are interpreted in integration, this one leading to integration by parts. The calculation of mean values is given as an example of the use of the definition. A chapter follows on general methods of finding indefinite integrals. Chapter three extends the notion in some directions, as that of improper integrals, line integrals, double and triple integrals, leading up to the formulas of Green and Ostrogradsky, although singularly the complete formula of Stokes is omitted at the point where it would have naturally come in. A chapter is given to elliptic functions, one to Fourier series, one to geometric applications, one to applications to mechanics. In the latter the notions of vector fields, curl, divergence, flux, vector lines and tubes, work, circulation, level surfaces, and potential are brought in. The approximate calculation of integrals has a chapter. The last two chapters consider differential equations, total and partial, and though quite elementary, the author has nevertheless given the student a good working basis. The accompanying problems of the manual of exercises reenforce the text materially.

The author states that he considers the needs of the student who has to study by himself, with little or no assistance. This no doubt accounts for the very plain treatment, and would suggest that in general a text written for students will be clearer than one written for the use of teachers. Of course a certain amount of rigor in books written for teachers is demanded, but rigor does not always lead to usable knowledge. Professor Zoretti is to be congratulated on his success.

JAMES BYRNIE SHAW.

College Algebra with Applications. By E. J. WILCZYNSKI. Edited by H. E. SLAUGHT. Boston, Allyn and Bacon, 1916. xx+507 pp.

THIS algebra, unlike the traditional college algebra, possesses unity, the centralizing theme being the function concept. The book opens with an excellent chapter on the number system of algebra, which is developed by means of the geometry of directed line segments. This chapter is followed by chapters on linear functions and progressions, quadratic functions and equations, integral rational functions of the nth order, fractional rational functions, irrational functions, the exponential function and logarithms, linear functions of more than one variable, permutations and combinations, probability, determinants, quadratic functions of two independent variables and simultaneous quadratic equations, sequences and series with a finite number of terms, limits and infinite series. There is an appendix which contains a short table of logarithms and a mortality table.

Throughout the book much emphasis is laid on the applications of algebra to problems in physics and chemistry. Such topics as the measurement of time, length and mass; the theory of the vernier and slide rule, logarithmic paper and scales; the notions of velocity, acceleration, density, specific gravity, force, uniform motion, etc., are fully discussed in the text, thus making it unnecessary for a student to have had a course in physics and chemistry.

The author's explanations are very lucid, but I wish to call particular attention to his treatment of complex numbers, partial fractions, and the notorious stumbling block "mathematical induction," where by means of some well-chosen examples he drives home the fact that both parts of the proof are equally important.

The author has introduced some analytics and some calculus in order to weave his subject matter about the function concept. He defines the slope of a line, finds the equations of lines, defines and derives the derivative of a rational integral function, applies derivatives to Taylor's expansion, maxima and minima, the finding of approximate irrational roots (Newton's method), etc.

Throughout the book appear many historical references which add greatly to the interest of the subject matter.

The reviewer believes that had the author omitted a few of the interesting but less important topics, and had he inserted more exercises, but preserved the unity which he has so carefully worked out, his book would serve better the needs of the average American college. This is an admirable book of reference and should be owned by every progressive teacher of mathematics.

F. M. MORGAN.

Annuities and Amortization Tables. By PIERRE ZALDARI. New York, Bankers Encyclopedia Co., 1917. 350 pp. Price $10.00.

THIS notice is written after reading the page proofs of the book that was to come from the press on December 15.

The preface states that "the preparation of this book was suggested to the writer primarily by the more or less recent development of what may be called foreign financing in this country, and the greatly increased importance in the position which America has assumed as a factor in the general scope of international banking; and some of these tables and problems were prepared to meet the specific demands of the Federal Farm Loan Act in that section which provides that the Farm Loan Board 'shall prepare and publish amortization tables.'"

The book contains 70 pages in which are propounded and solved 104 problems that embrace the whole range of the intricacies that arise in finance, covering the subjects of compound interest, annuity or annual investment, amortization, loans—issued at par, issued at a rate different from par and refunded at par, refunded above par, refunded above par with lottery privileges or premiums, the determination of the income rate and calculation of parities. There are ten chapters in this first part of the book. They are notably clear and in a form to make them readable by any one concerned, without a knowledge of mathematics beyond that given in the high school. Several of the solutions are not the ordinary ones in vogue in this country, and are quite neat.

The seven tables, only sample pages of which are before the reviewer, are to occupy 280 pages. The type and arrangement make these tables notably easy for reference. They are adequate to the solution of each of the 104 problems by a single multiplication. The tables have a wider range of usefulness than any that careful search has revealed within the covers of a single volume. They have more rates of interest than other tables, going from $\frac{1}{4}$ of 1 per cent to 10 per cent by steps of $\frac{1}{8}$ of 1 per cent. The periods are from 1 to 100. Some of the tables have no counterpart in other sets of tables. The numbers given in the tables carry from seven to eleven decimal places. Such a set of tables cannot but be received well by financiers in all lines of finance.

CHARLES C. GROVE.

NOTES.

At the Pittsburgh meeting of the American association for the advancement of science, assistant professor G. D. Birkhoff, of Harvard University, was elected vice-president of Section A, and Professor F. R. Moulton, of the University of Chicago, was reelected secretary of the Section.

The March number (series 2, volume 19, number 3) of the *Annals of Mathematics* contains the following papers: "Comparison theorems for homogeneous linear differential equations of general order," by R. D. Carmichael; "Note on a substitute for Duhamel's theorem," by H. B. Fine; "Some properties of a straight line and circle and their associated parabolas," by J. H. Weaver; "Motions in hyperspace," by C. L. E. Moore; "A definition of sense on closed curves in nonmetrical plane analysis situs," by J. R. Kline; "Covariant expansion of a modular form," by O. E. Glenn; "The intersections of a straight line and hyperquadric," by G. M. Green; "Numerical functions of $[x]$," by E. T. Bell; "Surfaces which can be generated in more than one way by the motion of an invariable curve," by L. P. Eisenhart.

The following mathematical papers appeared in volume 52 (1916–1917) of the *Proceedings of the American Academy of Arts and Sciences:* "Differential geometry of two-dimensional surfaces in hyperspace," by E. B. Wilson and C. L. E. Moore, number 6, pages 267–368; "A classification of quadratic vectors," by F. L. Hitchcock, number 7, pages 369–454.

At the meeting of the London mathematical society held January 17, 1918, the following papers were read: By P. A. MacMahon, "A method for studying any convergent series"; by G. H. Hardy, "Additional note on Dirichlet's division problem"; by J. H. Grace, "Note on a Diophantine approximation"; by K. A. Rau, "A note on a theorem of Mr. Hardy's"; by C. H. Forsyth, "Supernormal curves"; by H. Hilton and Miss D. S. Tuck, "Plane quartic curves with a tacnode."

THERE are probably many libraries throughout the country which contain the two volumes of "Correspondence of Scientific Men of the Seventeenth Century . . ." edited by Rigaud and published by the Oxford University Press in 1841. In 1862 a 36-page table of contents and index, compiled by Augustus De Morgan, was also issued (see Budget of Paradoxes). It is learned that these extra pages may still be purchased from the Oxford University Press, for forty cents.

THE following university courses in mathematics are announced for the summer session:

CORNELL UNIVERSITY (July 8 to August 16).—By Professor W. B. CARVER; Geometry for teachers, three hours.—By Professor W. A. HURWITZ: Algebra for teachers, three hours.—By Professor F. W. OWENS: Projective geometry, three hours.

UNIVERSITY OF CALIFORNIA (June 24 to August 3, at Los Angeles).—By Professor C. A. NOBLE: Analytic geometry, five hours; Mathematics for teachers, five hours.—By Professor VIRGIL SNYDER (of Cornell University): Calculus, five hours; Higher geometry, five hours.

UNIVERSITY OF CHICAGO (June 17 to August 31).—By Professor E. H. MOORE: Differential equations in general analysis (first half), four hours.—By Professor G. A. BLISS: Calculus of variations, four hours; Higher plane curves, four hours.—By Professor L. E. DICKSON: Solution of numerical equations (first half), four hours; Determinants and symmetric functions (second half), four hours; Algebraic invariants, four hours.—By Professor H. E. SLAUGHT: Definite integrals, four hours.—By Professor A. C. LUNN: Units and dimensions, four hours; Electromagnetic theory, four hours.—By Professor J. W. A. YOUNG; Topics in geometry, four hours.—By Professor R. G. D. RICHARDSON (of Brown University): Theory of functions of a complex variable, four hours.—By Professor W. H. ROEVER (of Washington University): Descriptive geometry, five hours.

UNIVERSITY OF ILLINOIS (June 18 to August 9).—By Dr. A. J. KEMPNER: Advanced algebra.—By Professor J. B.

SHAW: Differential equations; Vector methods.—By Professor C. H. SISAM: Solid analytic geometry; Differential geometry.

UNIVERSITY OF WISCONSIN (June 24 to August 2).—By Mr. R. E. MOORE: Differential equations, introductory course, five hours.—By Professor E. B. SKINNER: Differential geometry, three hours.—By Professor H. W. MARCH: Mechanics of a rigid body, five hours; Theory of probability, three hours.—By Dr. T. M. SIMPSON: Equations of the third and higher degrees, three hours.—By Professor E. B. VAN VLECK: Theory of the linear substitution in n variables, three hours; A survey of geometry, algebra, and trigonometry, treating of their development and history, five hours.

THE Academy of Sciences of Vienna has awarded the Baumgartner prize to Professor A. EINSTEIN.

PROFESSOR PAUL PAINLEVÉ has been elected president of the Paris academy of sciences.

MR. C. HANNEVIG has donated 150,000 crowns as a memorial to N. H. ABEL. The income of the fund is to be used to further mathematical research in Norway.

PROFESSOR L. CRELIER has been promoted to a full professorship at the University of Bern.

THE technical school at Vienna has conferred honorary doctorates on Professor R. MEHMKE, of the technical school at Stuttgart, and on Professor S. FINSTERWALDER, of the technical school at Munich.

PROFESSOR M. DISTELI, of the technical school at Carlsruhe, has retired, on account of ill health. Professor H. MOHRMANN, of the school of mines at Clausthal, has been appointed his successor.

PROFESSOR H. HAHN has been promoted to a full professorship of mathematics at the University of Bonn. DR. A. BECK has been appointed associate professor of mathematics.

Professor E. Schmidt, of the University of Breslau, has been appointed professor of mathematics at the University of Berlin.

The following university and college teachers of mathematics have recently entered the national military service:

Mr. J. E. Davis, of Pennsylvania State College, has joined the national army. Mr. J. J. Tanzola, of the U. S. Naval Academy, has joined the national army. Mr. H. M. Terrill, of Columbia University, has been made ensign in the naval reserve. Mr. A. L. Wechsler, of Columbia University, has joined the national army. Professor V. H. Wells, of the University of Pittsburgh, has been made lieutenant in the science and research division of the signal reserve corps. Mr. Frederick Wood, of the University of Wisconsin, has been appointed lieutenant in the field artillery.

The Normal medal of the American Society of Civil Engineers has been awarded to Mr. B. F. Groat, formerly professor of mathematics and mechanics at the University of Minnesota and now a consulting engineer of Pittsburgh, for his paper "Chemihydrometry and its application to the precise testing of hydro-electric generators." The theoretical considerations involved are largely of a mathematical nature.

Professor A. G. Caris, of the department of mathematics, has been appointed acting president of Defiance College, Defiance, Ohio.

Professor P. F. Smith has been made acting director of the Sheffield Scientific School, Yale University, during the absence of Director R. H. Chittenden on government service.

Mr. M. F. Jordan and Mr. Q. Stauffer have been appointed instructors in mathematics at the University of Maine.

Professor Léon Autonne, of the University of Lyons, died January 12, 1916, at the age of fifty-seven years.

Professor G. Frobenius, of the University of Berlin, died August 3, 1917, at the age of sixty-seven years.

PROFESSOR FRANZ LONDON, of the University of Bonn, died February 17, 1917, at the age of fifty-four years. Mr. ALFRED HAMBURGER has presented the University with an endowment of 30,000 marks for a foundation for mathematical research, in memory of Professor London.

PROFESSOR CHRISTIAN HORNUNG, of Heidelberg University, Tiffin, Ohio, died January 31, 1918, at the age of seventy-three years.

PROFESSOR A. T. G. APPLE, of the department of mathematics and astronomy of Franklin and Marshall College, died February 5, 1918, at the age of fifty-eight years.

DR. E. W. PONZER, assistant professor of applied mathematics at Stanford University, died December 20, 1917.

DR. E. A. ENGLER, formerly professor of mathematics and descriptive geometry at Washington University, St. Louis, and for ten years president of Worcester Polytechnic Institute, died January 16, 1918, at the age of sixty-one years. Professor Engler had been a member of the Society since 1892.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAM (C.). See DESCARTES (R.).

BOREL (E.). Leçons sur les fonctions monogènes uniformes d'une variable complexe. Rédigées par G. Julia. (Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Emile Borel.) Paris, Gauthier-Villars, 1917. 8vo. 12+166 pp. Fr. 7.50

BOWSER (E. A.). College algebra for the use of academies, colleges and scientific schools. 12th edition. Boston, Heath, 1917. 8vo. 14+540 pp. $1.60

DESCARTES (R.). Œuvres de Descartes publiées par C. Adam et P. Tannery sous les auspices du Ministère de l'Instruction publique. Paris, Cerf. Tomes 1–11: Œuvres, 1893–1909. Tome 12: Vie et œuvres de Descartes, étude historique par C. Adam, 1910. 20+646 pp. [Tome 13]: Supplément, index générale, 1913. 108 pp. 4to. Fr. 310.00

JULIA (G.). See BOREL (E.).

TANNERY (P.). See DESCARTES (R.).

II. ELEMENTARY MATHEMATICS.

DELBRIDGE (C. L.). Delbridge wonderful interest tables, 1 day to 365 days. . . . St. Louis, Delbridge, 1916. Folio. 371 pp. $20.00; leather, $25.00

DUDENEY (H. E.). Amusements in mathematics. London, Nelson, 1917. 8+258 pp. 3s. 6d.

FORSYTH (C. H.). Logarithms and anti-logarithms. Ann Arbor, George Wahr, 1917. 46 pp. Cloth. $0.75

JANNELLI (G.). Studio sulla trisezione dell'angolo. Palermo, tipografia Pontificia, 1917. 4to.

LENNES (N. J.). See SLAUGHT (H. E.).

MEAD (C. DEW.). Experiment in the fundamentals; giving the results of tests made in the Cincinnati schools with two kinds of practice material. (School efficiency monographs.) Chicago, World Book Company, 1917. 8vo. 14+54 pp. $0.60

SHORTHOSE (—.). Commercial ready reckoner. 9th edition. London, Allen and Unwin, 1916. 4to. 5s.

SLAUGHT (H. E.) and LENNES (N. J.). Plane geometry with problems and applications. Revised edition. Boston and Chicago, Allyn and Bacon, 1918. 8vo. 8+310 pp.

WARNE (—.). Handy ready reckoner, with over 38,000 calculations. London, Warne, 1916. 18mo. 240 pp. 7d.

III. APPLIED MATHEMATICS.

ALGRIN (—.). Eléments de mécanique générale et appliquée. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 533 pp.

ANGUS (R. W.). The theory of machines. 2d edition. New York, McGraw-Hill, 1917. 340 pp. $3.00

AUSTIN (F. E.). Examples in battery engineering. Hanover, N. H., F. E. Austin, 1917. 90 pp. Cloth. $1.25

BOUASSE (H.). Construction, description et emploi des appareils de mesure et d'observation. Paris, Delagrave, 1917. 8vo. 629 pp. Fr. 25.00

BOYD (J. E.). Strength of materials. 2d edition. New York, McGraw-Hill, 1917. 380 pp. $3.00

BRADLEE (S. G.). Stock brokers' daily balance interest table, for rapid computation on a 360 day basis . . . adapted to all classes of business where interest is figured on any amount up to $1,000,000 at every quarter per cent from 1 to 7 inclusive. Boston, Financial Publishing Company, 1916. 8vo. 95 pp. $7.50

CHRISTIE (C. V.). Electrical engineering. The theory and characteristics of electrical circuits and machinery. 2d edition. New York, McGraw-Hill, 1917. 550 pp. $4.00

COX (D. W.). Cox's tariff, tonnage and price extensions, from pounds to gross tons, computed for every hundred pounds from 100 to 160,000, at every five cents from five cents to five dollars fifty cents per gross ton of 2,240 pounds. Harrisburg, Pa., Charter Publishing Company, 1917. 4to. 222 pp. $20.00

Desmolliens (M.). Cours de langue anglaise à l'usage des élèves ingénieurs. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 444 pp.

Eiffel (G.). Nouvelles recherches sur la résistance de l'air et l'aviation faites au laboratoire d'Auteuil. Paris, Dunod et Pinat, 1914. 4to. 8+406 pp. Atlas, 67 pp.+39 plates. Reliés. Fr. 50.00

Eyrolles (L.). Formulaire mathématique et technique. 8e édition. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 16mo. 188 pp.

Gabeaud (—.). Introduction mathématique aux sciences techniques de l'ingénieur. 2e édition. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 608 pp.

Gouard (E.) et Hiernaux (G.). Cours élémentaire de mécanique industrielle. Principes généraux, applications, exercices pratiques. Paris, Dunod et Pinat. 8vo. Tome 1, 2e édition, 1914. Tome 2, Réimpression, 1917. Tome 3, Réimpression, 1917. 386+360+182 pp. Cartonné. Fr. 4.50+4.50+2.50

Hiernaux (G.). See Gouard (E.).

Higbee (F. G.). The essentials of descriptive geometry. 2d edition. New York, Wiley, 1917. 6+218 pp. Cloth. $1.80

Izart (J.). Aide-mémoire de l'ingénieur mécanicien. Recueil pratique de formules, tables et renseignements usuels. Réimpression. Paris, Dunod et Pinat, 1917. 8vo. 28+854 pp. Cartonné. Fr. 15.00

Jacquet (A.). Aide-mémoire de l'ouvrier mécanicien. Paris, Dunod et Pinat, 1917. 8vo. 174 pp. Cartonné. Fr. 3.50

Maillot (L.). Le manuel du mécanicien. Paris, Dunod et Pinat, 1917. 8vo. 8+173 pp. Cartonné. Fr. 2.50

Mills (J.). Radio communication. Theory and methods. With an appendix on transmission over wires. New York, McGraw-Hill, 1917. 11+205 pp. Cloth. $1.75

Peabody (C. H.). Naval architecture. 4th edition. New York, Wiley, 1917. 7+641 pp. Cloth. $7.50

Riccardo-Brauzzi (—.). Cours d'aéronautique générale. 2e partie: Etude statique et dynamique des ballons en général. Paris, Dunod et Pinat, 1917. 8vo. 410 pp. Fr. 30.00

See (T. J. J.). Electrodynamic wave-theory of physical forces. Volume 1. (Bulletins 1–6 inclusive.) Lynn, Mass., T. P. Nichols, 1917. 4to. 170 pp. $5.00

Spangler (H. W.). Graphics. New York, Wiley, 1917. 95 pp. Cloth. $1.25

Todd (J. A.). The mechanism of exchange; a handbook of currency, banking and trade in peace and in war. Oxford, Clarendon Press, 1917. 14+255 pp.

Willis (S. J.). A short course in elementary mathematics and their application to wireless telegraphy. London, Wireless Press, 1917. 3s. 6d.

Woodhouse (H.). Text-book of naval aeronautics. London, Laurie, 1917. Royal 8vo. 35s.

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and ninety-sixth regular meeting of the Society was held in New York City on Saturday, February 23, extending through the usual morning and afternoon sessions. The attendance included the following seventeen members:

Dr. F. W. Beal, Professor F. N. Cole, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. R. Hollcroft, Dr. L. L. Jackson, Mr. S. A. Joffe, Dr. J. R. Kline, Professor E. J. Miles, Professor R. L. Moore, Mr. George Paaswell, Dr. J. F. Ritt, Dr. Caroline E. Seely, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams.

Professor H. S. White occupied the chair. The Council announced the election of the following persons to membership in the Society: Miss M. F. Chadbourne, Smith College; Mr. Mervyn Davis, Equitable Life Insurance Company of Iowa; Mr. T. C. Fry, Western Electric Company; Dr. J. E. McAtee, University of Illinois; Dr. Norbert Wiener, Albany, N. Y. Four applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. J. F. Ritt: "Proof of the multiplication formula for determinants by means of linear differential equations."

(2) Dr. Olive C. Hazlett: "On vector covariants."

(3) Dr. P. R. Rider: "On the problem of the calculus of variations in n dimensions."

(4) Dr. A. R. Schweitzer: "On the iterative properties of an abstract group."

(5) Dr. A. R. Schweitzer: "On certain articles on functional equations."

(6) Dr. A. R. Schweitzer: "On iterative functional equations of the distributive type."

(7) Dr. J. R. Kline: "A new proof of a theorem due to Schoenflies."

(8) Professor R. L. Moore: "A sufficient condition that a system of arcs should constitute a surface."

(9) Mr. J. L. Walsh: "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function."

(10) Professor O. E. GLENN: "Covariant expansion of a modular form."

(11) Dr. J. F. RITT: "Polynomials with a common iterate."

(12) Professor L. P. EISENHART: "Transformations of applicable conjugate nets of curves on surfaces."

(13) Professor S. E. SLOCUM: "The romantic aspect of numbers."

Mr. Walsh's paper was communicated to the Society through Professor Bôcher. In the absence of the authors the papers of Dr. Hazlett, Dr. Rider, Dr. Schweitzer, Mr. Walsh, Professor Glenn, Professor Eisenhart, and Professor Slocum were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Ritt's paper, which is presented only as an interesting piece of manipulation, shows how the formula for the multiplication of determinants can be found from the properties of Wronskian determinants.

2. This paper concerns itself with a certain type of covariant of the general linear algebra which, for convenience, is called a vector covariant. In a previous paper, Dr. Hazlett has developed the theory of a rather different type of covariant, referred to here as scalar covariants, since they involve only the coordinates x_i of the general number of the algebra $X = \Sigma x_i e_i$ and the constants of multiplication γ_{ijk} $(i, j, k = 1, \ldots, n)$ in contradistinction to the vector covariants which involve the units e_i $(i = 1, \ldots, n)$ in addition to the x's and γ's. Since multiplication of the units in general is neither commutative nor associative, one might anticipate some rather inconvenient difficulties. These can, however, be avoided by a device. Accordingly, the annihilators of vector covariants are found and hence it is shown that every rational integral vector covariant of the n-ary linear algebra is a covariant of the general number $X = \Sigma x_i e_i$ of the algebra and a suitable set of scalar covariants of the algebra. From this fact flow theorems analogous to those proved for scalar covariants and and, in particular, the "finiteness" of vector covariants.

3. Dr. Rider's paper considers the theory of minimizing an

integral of the form

$$\int_{t_0}^{t_1} f(x_1, \cdots, x_n, \tau_1, \cdots, \tau_{n-1}) \sqrt{x_1'^2 + \cdots + x_n'^2}\, dt,$$

in which

$$\tau_i = \arctan x'_{i+1}/\sqrt{x_1'^2 + \cdots + x_i'^2}.$$

The Euler equations are derived and the transversality and corner conditions are given, as are the forms of the Hilbert invariant integral and the Weierstrass e-function. Certain necessary conditions, including the condition of Weierstrass, are obtained. The paper will be published in the *Tôhoku Mathematical Journal.*

4. Dr. Schweitzer gives the following postulates* for an Abelian group: I_1. x, y imply $\phi(x, y) = z$ uniquely. I_2. x, y imply $f(x, y) = z$ uniquely. II_1. $\phi\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$. II_2. $\phi\{f(x, y), x\} = x$. II_3. $f\{\phi(x, y), x\} = x$. II_4. $\phi(x, y) = \phi(y, x)$ if $x \neq y$. The author shows that if II_4 is replaced by II_4', $\phi(y, y') = \phi(y, y)$ implies $y = y'$ if $y \neq y'$, then the postulates are satisfied by any group, finite or infinite. In the latter set postulate II_1 may be replaced by II_1', $f\{f(y, x), f(z, x)\} = f(y, x)$, without loss of generality.

In the second part of the paper, on the basis of the preceding postulates, the author points out concepts in group theory expressible in terms of iterative compositions, such as the transform of an element, the commutator of two elements. In particular, infinite sequences of iterative compositional relations are obtained which are satisfied by the elements of any group. Such relations are, e. g., of the type $\psi\{u_1^{(i)}, u_2^{(i)}, \cdots, u_n^{(i)}\} = \psi\{x_1, x_2, \cdots, x_n\}$ · where $u_i^{(1)} = \psi\{x_i, t_1, t_2, \cdots, t_{n-1}\}$, $u_i^{(2)} = \psi\{t_1, x_i, t_2, \cdots, t_{n-1}\}$ etc., and $i = 1, 2, \cdots, n$ $(n = 2, 3, \cdots)$.

The preceding postulates are free from explicit statements of existence. In the *Transactions* (1905) Dickson in effect eliminates the explicit existential properties of the Moore-Dickson postulates for a finite group.

5. Dr. Schweitzer's second paper is a constructive criticism of certain articles† on functional equations due to Suto on the

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* Cf. Grassmann, Gesammelte Werke, vol. I_2, p. 239.

† *Tôhoku Math. Journal,* vol. 3 (1913), vol. 6 (1914–1915).

basis of the author's independent investigations. Numerous errors of Suto are pointed out; in particular the methods by which the iterative functional equation, loc. cit., volume 3, page 60, is solved are erroneous. This equation is but one of an important class of iterative equations previously defined by the author, e. g.,

$$f\{u_1, u_2, \cdots, u_n\} = f\{v_1, v_2, \cdots, v_n\} \quad (n = 2, 3, \cdots),$$

where

$$u_i = f\{x_{1i}, x_{2i}, \cdots, x_{ni}\}, v_i = f\{x_{j_1k_1}, x_{j_2k_2}, \cdots, x_{j_nk_n}\}$$

and $x_{j_1k_1}$ is some one of the arguments of u_1, $x_{j_2k_2}$ is some one of the arguments of u_2, etc., and $i = 1, 2, \cdots, n$. The equation treated by Suto was solved by the author by direct differentiation. Concerning the "associative" functional equations the author obtains a number of theorems the simplest of which is: If $\phi\,\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$ and $\phi(x', y) = z$ implies $x' = f(z, y)$ then $\phi(x, y) = \psi^{-1}\{\psi(x) + \psi(y)\}$ and $f(x, y) = \psi^{-1}\{\psi(x) - \psi(y)\}$. The equation of Suto, loc. cit., volume 6, page 82, is a special case of the equation

$$\sum_{i=1}^{n} f_i(x, y) = \psi(x, y),$$

where the functions $f_i(x, y)$ satisfy given partial differential equations of the first order and $\psi(x, y)$ is a given "quasi-transitive" or "symmetric" function, e. g., $x - y$, x/y, $x + y$, $x \cdot y$. In particular important functional equations arise if the $f_i(x, y)$ are required to satisfy given iterative functional equations. Finally Suto's paper on the arithmetic mean, loc. cit., volume 6 (page 79), is compared with the author's.

6. Dr. Schweitzer proves the following theorem: A sufficient condition that

$$\begin{aligned} &\phi\{\phi(x_{11}, \cdots, x_{1n}), t_1, t_2, \cdots, t_{n-1}\} \\ (1) \qquad &\qquad = \phi\{\phi(x_{11}, t_1, \cdots, t_{n-1}), \cdots, \phi(x_{1n}, t_1, \cdots, t_{n-1})\}, \end{aligned}$$

where $n = 2, 3, \cdots$, is that

$$\begin{aligned} &\phi\{\phi(x_{11}, \cdots, x_{1n}), \cdots, \phi(x_{n1}, \cdots, x_{nn})\} \\ (2) \qquad &\qquad = \phi\{\phi(x_{11}, \cdots, x_{n1}), \cdots, \phi(x_{1n}, \cdots, x_{nn})\} \end{aligned}$$

and

$$\phi\{\phi(x_{11}, \cdots, x_{1n}), \phi(t_1, \cdots, t_1), \cdots, \phi(t_{n-1}, \cdots, t_{n-1})\}$$
$$= \phi\{\phi(x_{11}, \cdots, x_{1n}), t_1, t_2, \cdots, t_{n-1}\}$$

and consequently infers a solutions of (1). The result is generalized to iterative equations of infinite degree.* The equation (1) is a distributive equation of order 1; equation (2) is a limiting case, viz., of order n. Equations of order k, $1 \leqq k \leqq n$ are readily defined, e. g.,

$$\phi\{\phi(x_1, x_2, x_3)\phi(y_1, y_2, y_3)t\} = \phi\{\phi(x_1, y_1, t)\phi(x_2, x_2, t)\phi(x_3, y_3, t)\}$$

is distributive of the second order. Extension to distributive iterative equations on two functions is made.

7. In a paper, presented to the Society at the December meeting, Professor R. L. Moore proved the following theorem: If the points A and C separate B and D on the simple closed curve J, then it is possible to draw a set of simple continuous arcs such that through every point of J and its interior there are two and only two arcs of the set, the arcs AB, BC, CD and DA being arcs of the set. This theorem was proved on the basis of his set of axioms Σ_1†. With the use of this theorem Dr. Kline gives a simple proof of the following theorem, due to Schoenflies: Given two simple closed plane curves J_1 and J_2, which are in one-to-one continuous correspondence under a correspondence Π. Let R_i $(i = 1, 2)$ denote the point set composed of J_i plus I_i, the interior of J_i. Then there exists a continuous one-to-one correspondence between the points of R_1 and R_2 such that the points of J_1 and J_2 correspond as fixed by the correspondence Π. For an abstract of a proof of this theorem by Dr. G. A. Pfeiffer from a somewhat different point of view see this BULLETIN, volume 24, number 4, page 171.

8. Suppose that AB and CD are two arcs in three-dimensional space. Suppose that G is a bounded self-compact system of simple continuous arcs such that each arc of G has its end points on AB and CD respectively and no two arcs of

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* Cf. Suto, loc. cit., vol. 6 (1914–1915), p. 16.

† Cf. R. L. Moore, "On the foundations of plane analysis situs," *Transactions Amer. Math. Society*, volume 17 (1916), pp. 131–164.

G have any point in common. Professor Moore proposes to show: (1) that the point set constituted by such a system of arcs cannot always be brought into continuous one-to-one correspondence with the surface consisting of a circle plus its interior; (2) that such a correspondence will exist in case the arcs of the system G are equicontinuous. He has under consideration further theorems of this type.

9. Mr. Walsh's paper will be published in the *Transactions*.

10. Professor Glenn's paper contains elaboration, and computations based on particular cases, of a former theorem of his concerning the reducibility (mod p) of the general binary quantic. The forms of orders $> p^2-1$ ($p = 2, 3$) are reducible modulo p in terms of modular covariants of orders 0 to p^2-1, as polynomials or covariant expansions in the universal covariants. Such an expansion is a form of generalized modular typical representation in which the multiplier concomitant is unity. Tables are given showing the expansions and all constituent covariants occurring in them, for forms of the first eleven orders modulo 2, and for the orders 9, 10, 11 when the modulus is 3.

The paper appeared in the March, 1918, number of the *Annals of Mathematics*.

11. Dr. Ritt determines the conditions under which two different polynomials $F(x)$ and $f(x)$, both of degree n greater than one, can have a common pth iterate. $F(x)$ can be derived by means of a linear integral transformation from a polynomial $g(x) = x^n + k_1x^{n-q} + k_2x^{n-2q} + \cdots k_rx^{n-rq}$, where q is some divisor of $(n^p - 1)/(n - 1)$, and $f(x)$ will be derived by the same transformation from $\omega g(x)$, where ω is a qth root of unity. The proof is accomplished with the help of an important theorem on iteration due to L. Boettcher, which has lain hidden for a long time in a paper published in Russian.*

12. When two surfaces are applicable to one another, in the correspondence thus established there is a unique conjugate system of curves on each surface corresponding to a conjugate system on the other. These two conjugate systems, or nets, are said to be applicable nets. Let N and $\bar{N}$ be two such applicable nets. Peterson showed that when a net N', parallel

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* *Bulletin of the Kasan Math. Society*, vol. 14 (1905) p. 176.

to N, is known, a net $\bar{N}'$, parallel to $\bar{N}$, can be found by quadratures such that N' and $\bar{N}'$ are applicable. Professor Eisenhart has applied to applicable nets the theory of transformations T (cf. *Transactions*, volume 17, page 97), and has established transformations of applicable nets into applicable nets. The transformations of nets admitting an infinity of applicable nets into similar nets are of particular interest. The paper will be published in the April number of the *Transactions*.

13. Professor Slocum's paper calls attention to the important rôle played by number symbolism in the intellectual development of mankind. Wherever mind has reacted to the stimulus of natural phenomena, the number concept has resulted as the inevitable expression of the laws governing the material universe. The properties of number, first accepted as a fact, subsequently came to be regarded as symbolic, and only in recent times has the mind been able to grasp their true significance as one aspect of the great principle of functionality.

Various instances are cited showing that the ancient customs of all races imply a distinct belief in the action of numbers on the course of human events. Beginning with the empiricism of primitive nations, like the Egyptians, Persians, and Greeks, it is shown that as civilization developed number ideas became symbolic, and the properties of numbers were invested with physical attributes. Numerals were regarded not merely as passive symbols but as active principles of good and evil, capable of reproducing their properties in accordance with mathematical operations. The extensive use of number symbolism by the ancient Hebrews, however, stands in marked contrast to that characteristic of pagan nations like the Greeks. With the Hebrews number symbolism was simply incidental to their rich Oriental imagery, and there is no evidence anywhere in Old Testament literature that the properties of numbers were even known, much less regarded as exerting any mystic influence on human destiny. But when the New Testament was written the Hebrews had outgrown their isolation, and Grecian influence is clearly apparent in the mystic character given to number symbolism, especially in the Apocalypse.

During the middle ages the attempt of scholasticism to reconcile the tenets of Christianity with Grecian philosophy lent special impetus to the study and development of sacred hermeneutics. The growth of this peculiar by-product of scholasticism is outlined, with references to the work on number symbolism of the Venerable Bede, St. Augustine, and numerous other prominent ecclesiastics. In particular, a remarkable example is given of a ninth century rendering of the account given in Genesis 18 of the conflict of the 318 servants of Abraham against the four kings. The classic example of the number of the beast is also studied, and it is shown that its interpretation serves to establish the date of the writing of Revelation, long in dispute, as well as affords a key to the system of number symbolism used by St. John.

F. N. COLE,
Secretary.

SOME REMARKABLE DETERMINANTS OF INTEGERS.

BY PROFESSOR E. T. BELL.

1. THE determinants in this note are arithmetical rather than algebraic in character; their properties, not obvious by the usual reductions, follow immediately from simple considerations in the theory of numbers. Throughout, letters other than x and functional signs denote positive integers, and $[x]$ is the greatest integer in x.

2. Let $D_n\,|F(k),\ G(k)\,|$ denote the determinant of the nth order whose first and last columns are respectively $F(1)$, $F(2)$, $\cdots$, $F(n)$, and $G(1)$, $G(2)$, $\cdots$, $G(n)$; and whose $(1+k)$th column $(k = 1, 2, \cdots, n-2)$ is derived from the first by prefixing k zeros and repeating in succession each element of the first $(1+k)$ times, until in all a column of n elements has been written down

$$(1)\qquad D_n\,|\,F(k),\ G(k)\,| \equiv \begin{vmatrix} F(1) & 0 & 0 & \cdots & G(1) \\ F(2) & F(1) & 0 & \cdots & G(2) \\ F(3) & F(1) & F(1) & \cdots & G(3) \\ F(4) & F(2) & F(1) & \cdots & G(4) \\ F(5) & F(2) & F(1) & \cdots & G(5) \\ F(6) & F(3) & F(2) & \cdots & G(6) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{vmatrix}.$$

For a reason appearing presently, $D_n|G(k), F(k)|$ is called the inverse of (1). A genesis of (1) is as follows: Let

$$(2)\qquad G(n) = \sum_{a=1}^{n} g(a)F([n/a]);\quad F(1) = 1;\quad F(0) = 0.$$

Then, putting in (2) successively $n = 1, 2, \cdots, n$, and solving the resulting system for $g(n)$, we find $g(n) = D_n|F(k), G(k)|$.

3. The more interesting determinants (1) relate to $F \equiv f'$, $G \equiv \overline{gf}'$, where the notation is

$$(3)\quad f'(n) \equiv \sum_{a=1}^{n} f(a);\ \overline{gf}(n) \equiv \sum_{n} g(d)f(n/d);\ \overline{gf}'(n) \equiv \sum_{a=1}^{n} \overline{gf}(a),$$

the $\sum_{n}$ extending to all divisors $1, d, \cdots, n$, of n. Clearly, $\overline{gf}(n) = \overline{fg}(n)$, and $\overline{gf}'(n)$ denotes $\Sigma g(d)f(\delta)$, the Σ referring to all d, δ such that $d\delta \leqq n$; whence, noting the number of times that d takes the particular value a, obviously

$$(4)\qquad \overline{gf}'(n) = \sum_{a=1}^{n} g(a)f'([n/a]) = \sum_{a=1}^{n} f(a)g'([n/a]).$$

Comparing (4), (2), we have, for $f(1) = g(1) = 1$, and therefore $f'(1) = g'(1) = 1$

$$(5)\quad g(n) = D_n|f'(k), \overline{gf}'(k)|;\ f(n) = D_n|g'(k), \overline{gf}'(k)|.$$

4. To limit the paper, henceforth g, f denote ψ-functions, viz., solutions of $\psi(1) = 1$, $\psi(mn) = \psi(m)\psi(n)$ for m, n relatively prime. It may be shown without difficulty that when g is given, there always exists a unique ψ-function, g_1, such that $\overline{gg_1}(1) = 1$, and $\overline{gg_1}(n) = 0$ for $n > 1$. We shall call such g, g_1 reciprocals of each other; and it is easily verified that for g, g_1 and f, f_1 reciprocals, (4) implies the inversions

$$(6)\quad f'(n) = \sum_{a=1}^{n} g_1(a)\overline{gf}'([n/a]);\ g'(n) = \sum_{a=1}^{n} f_1(a)\overline{gf}'([n/a]);$$

whence in all, as consequences of (4), the pairs of inverses

$$(7)\quad \left.\begin{aligned} f(n) &= D_n|g'(k), \overline{gf}'(k)| \\ f_1(n) &= D_n|\overline{gf}'(k), g'(k)| \end{aligned}\right\};\quad \left.\begin{aligned} g(n) &= D_n|f'(k), \overline{gf}'(k)| \\ g_1(n) &= D_n|\overline{gf}'(k), f'(k)| \end{aligned}\right\}.$$

Again, from (7) and the definitions directly, if $\overline{gf}(n) = h(n)$

$$(8)\qquad h(n) = \sum_{n} \{D_d\,|\,g'(k),\,\overline{gf}'(k)\,|\cdot D_{n/d}\,|\,f'(k),\,\overline{gf}'(k)\,|\};$$

and, noting that f, f_1 and g, g_1 are reciprocals, we have for $n > 1$

$$(9)\qquad 0 = \sum_{n} \{D_d\,|\,g'(k),\,\overline{gf}'(k)\,|\cdot D_{n/d}\,|\,\overline{gf}'(k),\,g'(k)\,|\},$$

and the like with g, f interchanged. Also, for m, n relatively prime (but not otherwise)

$$(10)\quad D_{mn}\,|\,f'(k),\,\overline{gf}'(k)\,| = D_m\,|\,f'(k),\,\overline{gf}'(k)\,|\cdot D_n\,|\,f'(k),\,\overline{gf}'(k)\,|,$$

and similarly with g, f interchanged. Hence, if $n = p^a \cdots r^b$ is the resolution of n into prime factors, $D_n\,|\,f'(k), \overline{gf}'(k)\,|$ = the product of similar determinants of respective orders $p^a, \cdots, r^b$.

5. At least one pair of the (7) will reduce to determinants of pure integers, viz., free from all functional signs f', g', $\overline{gf}'$, for such f, g as give one of $f'(k)$, $g'(k)$, and $\overline{gf}'(k)$ explicit functions of k. In the theory of numbers there are a great many such f, g: to illustrate § 4 we select in § 6 a few of the simplest. Again, for f, g given ψ-functions, it is known, or may be proved easily from first principles, that there always exists a unique ψ-function h, in general simpler than $\overline{fg}$, such that $\overline{fg}(n) = h(n)$. For specific f, g this reduction is readily effected when $n = p^a$, p prime, and hence immediately from the definition of ψ-functions for general n; similarly for the reciprocal f_1 of specific f. All the like reductions quoted in § 6 are either well known or may be thus verified with ease. We shall require the definition: a number which is divisible by no square > 1 is *simple*.

6. In illustration, consider the determinants dependent upon the following set of intimately related ψ-functions and their simpler properties: (i) $u_r(n) \equiv n^r$, whose reciprocal for $r > 0$, is $u_r(n)\mu(n)$. Also, u_0, μ are reciprocals; hence $\overline{u_0\mu}'(k) = 1$. Clearly, $u_0'(k) = k$; $u_1'(k) = \frac{1}{2}k(k+1)$. (ii) $\varpi(n) \equiv (-1)^{\lambda(n)}$, $\lambda(n)$ the *total* number of prime divisors of n. Writing $\mu(n) \times \mu(n) \equiv \mu^2(n)$; ϖ, μ^2 are reciprocals, whence

$$\overline{\varpi\mu^2}'(k) = 1.$$

Also $\overline{\mu^2 t_2} = u_0$, $\overline{u_0\varpi} = t_2$; whence $\overline{u_0\varpi}'(k) = t_2'(k) = [\sqrt{k}]$; $\overline{\mu^2 t_2}'(k) = u_0'(k) = k$. (iii) $\mu(n) = 0$ or $\varpi(n)$ according as n is not, or is, simple (Möbius' function). (iv) $t_r(n) = 1$ or 0 according as n is, or is not, an rth power > 1. The reciprocal is $t_r(n)\mu(\sqrt[r]{n})$; and $t_r'(k) = [\sqrt[r]{k}]$.

We apply the $f(n)$, $f_1(n)$ pair in (7) to these. For $f \equiv \mu$, $g \equiv u_0$, by (i)

$$(11) \qquad \mu(n) = D_n|k,\, 1|; \quad 1 = D_n|1,\, k|.$$

For $f \equiv \mu^2$, $g \equiv t_2$, (ii) gives

$$(12) \qquad \mu^2(n) = D_n|[\sqrt{k}],\, k|; \quad \varpi(n) = D_n|k,\, [\sqrt{k}]|.$$

From the reciprocals (by (i)), $f(n) \equiv n\mu(n)$, $g(n) \equiv u_1(n)$,

$$(13) \quad n\mu(n) = D_n|\tfrac{1}{2}k(k+1),\, 1|; \quad n = D_n|1,\, \tfrac{1}{2}k(k+1)|.$$

To complete this set we add $\varphi_r(n)$ and Gegenbauer's $\mu_r(n)$; (v) $\mu_r(n) = 0$ if n is divisible by an rth power $(r > 1)$, otherwise $= 1$. The reciprocal, γ_r, is defined by $\overline{\mu t_r} = \gamma_r$. Also, $\overline{\mu_r t_r} = u_0$; $\mu_2 = \mu^2$. (vi) $\varphi_r(n) =$ the number of integers $\leq n^r$ that are divisible by the rth power of no prime divisor of n^r; $\varphi_1 \equiv \varphi$, the ordinary totient. The reciprocal, χ_r, is defined by $\chi_r(p^a) = (1 - p^r)$; also $\overline{u_0\varphi_r} = u_r$.

Whence, similarly to (11)–(13), we have for $f \equiv u_r$, $g \equiv t_r$; and for $f \equiv \varphi_r$, $g \equiv u_0$,

$$(14) \qquad \mu_r(n) = D_n|[\sqrt[r]{k}],\, k|; \quad \gamma_r(n) = D_n|k,\, [\sqrt[r]{k}]|.$$

$$(15) \qquad \begin{aligned} \varphi_r(n) &= D_n|k,\, (1^r + 2^r + \cdots + k^r)|; \\ \chi_r(n) &= D_n|(1^r + 2^r + \cdots + k^r),\, k|. \end{aligned}$$

For $r = 2$, (14) becomes (12); and for $r = 1$, writing $\chi_1 \equiv \chi$, (15) gives

$$(16) \quad \varphi(n) = D_n|k,\, \tfrac{1}{2}k(k+1)|; \quad \chi(n) = D_n|\tfrac{1}{2}k(k+1),\, k|.$$

The theory of numbers furnishes a ready means of writing down any desired quantity of such results. Enough have been given for the purpose of showing how, combined with the definitions of specific ψ's, we may, as in § 7, deduce properties of the determinants themselves.

7. One class of relations between the D_n is obtained on

substituting in (8), (9), (10). E. g., by (10), (11) for m, n relatively prime: $D_{mn}|k, 1| = D_m|k, 1|\cdot D_n|k, 1|$; etc. As examples of another, we have obviously $\mu^{2s+1}(n) = \mu(n)$, $\mu^{2r}(n) = \mu^2(n)$; whence from (11), (12)

$$(17) \qquad \begin{aligned} D_n{}^r|[\sqrt{k}], k| &= D_n|[\sqrt{k}], k| = D_n{}^2|k, 1|; \\ D_n{}^{2s+1}|k, 1| &= D_n|k, 1|. \end{aligned}$$

Again, from (11), (13)

$$(18) \qquad D_n|1, \tfrac{1}{2}k(k+1)|\cdot D_n|k, 1| = D_n|\tfrac{1}{2}k(k+1), 1|.$$

These suffice for this kind. Curious results are found when n is prime; we illustrate, however, another kind of relation which gives less obvious properties of the D_n. Let the order n be a simple number s (defined, § 5). Then, it follows readily from the definitions of the functions that $\gamma_r(s) = \varpi(s) = \mu(s)$; $\mu_r(s) = 1$, and $\varphi_r(s) = \varpi(s)\chi_r(s)$. Hence, from (14), (12), (11)

$$(19) \qquad D_s|k, [\sqrt[r]{k}]| = D_s|k, [\sqrt{k}]| = D_s|k, 1|,$$

$$(20) \qquad D_s|[\sqrt[r]{k}], k| = 1 = D_s|1, k|;$$

and from (15), (12)

$$(21) \qquad \begin{aligned} &D_s|k, (1^r + 2^r + \cdots + k^r)| \\ &\qquad = D_s|k, [\sqrt{k}]|\cdot D_s|(1^r + 2^r + \cdots + k^r), k|, \end{aligned}$$

which, combined with (19), (20), gives further s-identities. A totally distinct kind of result may be found by using the g, g_1 pair of (7) instead of, as above, the f, f_1 pair; the D_n in this case contain ψ-functions of k. Last, isomorphic to the ψ-functions there is a class of Ψ-functions having the same formal properties as the ψ's, but based upon the resolution of n into distinct (viz., relatively prime) simple numbers, $P_1, \cdots, P_r$ (thus, $n = P_1{}^a \cdots P_r{}^b$), instead of into distinct primes; and for these Ψ's there are theorems on determinants corresponding to those for the ψ's, including the kind (19) to (21), the correspondents in the latter case being for $S = P_1 \cdots P_r$.

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A THEOREM ON SEMI-CONTINUOUS FUNCTIONS.

BY PROFESSOR HENRY BLUMBERG.

(Read before the American Mathematical Society December 1, 1917.)

RECENTLY G. C. Young[*] and A. Denjoy[†] have communicated theorems—those in Denjoy's memoir are of an especially comprehensive character—dealing, in particular, with point sets where the four derivatives of a given continuous function are identical. It is the purpose of this note to treat an analogous problem that arises when "derivative" is replaced by "saltus."[‡] However, instead of confining ourselves to "saltus," we prove a more general theorem that applies essentially to all semi-continuous functions.[§] We preface the proof of this theorem with the following

LEMMA. *Let* $f_1(x)$ *and* $f_2(x)$ *be two real, single-valued functions, defined in the linear continuum, such that everywhere* $f_1(x) \geqq f_2(x)$, *and moreover, for every fixed real number* k, *the set* S_k *of points* x *where* $f_1(x) \geqq k$ *and* $f_2(x) < k$ *is countable. Then* $f_1(x)$ *and* $f_2(x)$ *are identical except at most in a countable set.*

Proof. Let $\{k_n\}$, $n = 1, 2, \cdots \infty$, be a set of k's everywhere dense in the linear continuum. The set[||] $S = \mathfrak{S}(S_{k_n})$, which consists of all of the elements of every S_{k_n}, is also countable. We show that $f_1(x) = f_2(x)$ for every given x not in S. For let $\{k_{i_n}\}$ be a monotone decreasing sequence of k_n's having $f_2(x)$ as limit. Since x is given as not belonging to S, it must be that $f_1(x) < k_{i_n}$ for every n; for from $f_1(x) \geqq k_{i_n}$ and $f_2(x) < k_{i_n}$, we would conclude that x belonged to $S_{k_{i_n}}$ and hence to S. It follows that $f_1(x) \leqq f_2(x)$, and hence, since $f_1(x) \geqq f_2(x)$ by hypothesis, that $f_1(x) = f_2(x)$.

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[*] *Acta Mathematica*, vol. 37 (1914), p. 141.

[†] *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 105.

[‡] By the saltus (= oscillation) of a given function $f(x)$ at the point x we understand the greatest lower bound of the saltus of $f(x)$ in the interval (ξ, η), for all intervals (ξ, η) enclosing x as interior point. Cf. Hobson, The Theory of Functions of a Real Variable (1907), art. 180, and the author's paper, "Certain general properties of functions," *Annals of Mathematics*, vol. 18 (1917), p. 147. For the comprehensiveness of our result see the remark at the close, in conjunction with the theorem of this note and the corollaries.

[§] "Essentially" in the sense that every semi-continuous function is exhibitable as the "associated" function $\phi(x)$ of a "monotone decreasing interval-function ϕ_{ab}" (see theorem below).

[||] In regard to the notation, compare Hausdorff, Grundzüge der Mengenlehre (1914), p. 5.

Definitions. ϕ_{ab} is said to be a "monotone, decreasing, real interval function,"[*] if ϕ_{ab} is a real number for every (closed) interval (a, b) and, in addition, $\phi_{\alpha\beta} \leqq \phi_{ab}$ if (α, β) is contained in (a, b). With every such interval function, we associate three point functions: (1) $\phi(x)$, the greatest lower bound of all ϕ_{ab} such that the fixed point x is in the interior of (a, b); (2) $\phi^{(r)}(x)$, the greatest lower bound of ϕ_{xb}, for all b's greater than x; and (3) $\phi^{(l)}(x)$, the greatest lower bound of ϕ_{ax} for all a's less than x.

The functions $\phi(x)$, $\phi^{(r)}(x)$ *and* $\phi^{(l)}(x)$ *associated with a monotone, decreasing, real interval function* ϕ_{ab} *that has a finite lower bound necessarily exist, and* $\phi(x)$ *is an upper semi-continuous function.* (For the proof, the reader is referred to a former paper by the author.)[†]

We now come to the

THEOREM. *Let* ϕ_{ab} *be a monotone, decreasing, real interval function having a finite lower bound. Then the three associated point functions* $\phi(x)$, $\phi^{(r)}(x)$ *and* $\phi^{(l)}(x)$ *(exist and) are identical except at most in a countable set.*

Proof. It is evidently sufficient to show that $\phi(x)$ and $\phi^{(r)}(x)$ are identical except in a countable set. From the definition of these functions, it follows that $\phi(x) \geqq \phi^{(r)}(x)$ for every x. For if (a, b) contains x as interior point, we have, according to the monotoneity property of ϕ, the inequality $\phi_{xb} \leqq \phi_{ab}$. We may thus associate with every number ϕ_{ab} of the set of which $\phi(x)$ is the greatest lower bound a smaller or equal number ϕ_{xb} of the set of which $\phi^{(r)}(x)$ is the greatest lower bound. Let now S_k represent, for a fixed real number k, the set of points x where simultaneously $\phi(x) \geqq k$ and $\phi^{(r)}(x) < k$.[‡] If x belongs to S_k, there must exist, on account of the relation $\phi^{(r)}(x) < k$, an interval (x, b) such that $\phi_{xb} < k$. From the last inequality, it follows that $\phi(\xi) < k$ for every ξ interior to (x, b); hence ξ does not belong to S_k. We have thus associated with every point x of S_k an interval (x, b) having no interior points belonging to S_k.[§] Clearly two such associated intervals cannot overlap. Since a set of non-overlapping intervals in the linear continuum is at most countable, it follows that S_k is at most countable. By the use of the lemma we now conclude, identifying $f_1(x)$ with $\phi(x)$ and $f_2(x)$ with $\phi^{(r)}(x)$, that $\phi(x)$ and $\phi^{(r)}(x)$ differ at most in a countable set.

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[*] Defined, as is the case with all the functions treated in this note, in the entire linear continuum.

[†] *Annals of Mathematics*, l. c.

[‡] Cf. G. C. Young, l. c., p. 143.

[§] This association may be made unique by taking the point b as far to the right as possible, and allowing, if necessary, the "ideal" interval (x, ∞).

If, in particular, ϕ_{ab} means the saltus of a given real function $f(x)$ in the interval (a, b), we obtain as associated point functions of ϕ the ordinary saltus and two other functions, which may be designated as the "right saltus" and the "left saltus." We thus have the

COROLLARY. *The three functions $\phi(x)$, $\phi^{(r)}(x)$, and $\phi^{(l)}(x)$, representing respectively the saltus, the right saltus and the left saltus of a given function $f(x)$ at the point x, are identical except at most in a countable set.*

The *f*, *d*, *e*, *n.d.* and *z*-saltus functions, which are the saltus functions when respectively finite, denumerable, exhaustible, non-dense sets and sets of zero measure may be neglected,* likewise yield particular cases of our theorem, and we have the

COROLLARY. *The f, d, e, n.d. and z-saltus functions are identical with the corresponding right and the left saltus functions except at most in a countable set.*

As a special case of our first corollary we have the

COROLLARY. *The points where a given function $f(x)$ is discontinuous but continuous on the right (left) constitute at most a countable set.*

Remark. The exhaustive character of our theorem may be judged from the fact that if $C = \{x_1, x_2, \cdots\}$ is any given countable set of real numbers, a function $f(x)$ exists whose saltus, f-saltus, e-saltus, etc., coincides everywhere respectively with its right saltus, right f-saltus, right e-saltus, etc., except precisely at the points of C. In fact,

$$f(x) = \sum_{x_n \leqq x} \frac{1}{2^n}$$

—i. e., the sum of all the fractions $1/2^n$ such that $x_n \leqq x$—is such a function. For it may be readily shown that $f(x)$ has its right saltus, right f-saltus, etc., everywhere 0, and its saltus, f-saltus, etc., everywhere 0 except at every point x_n of C, where these saltus functions all take the value $1/2^n$.

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* Cf. *Annals of Mathematics*, l. c.

SURFACES OF REVOLUTION IN THE THEORY OF LAME'S PRODUCTS.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society September 4, 1916.)

IN *Berliner Monatsberichte*, February, 1878, Wangerin discussed the problem of the most general orthogonal surfaces of revolution such that if Laplace's equation be written in coordinates corresponding to these surfaces a solution may be obtained in the form of a Lamé's product with an extraneous factor, i. e.,

$$V = \lambda \cdot R_1 \cdot R_2 \cdot \theta. \tag{1}$$

R_1, R_2, and θ are functions respectively of the parameters of the two families of surfaces and the meridian planes, while λ may involve all three parameters. Wangerin showed that λ is $1/\sqrt{r}$, where r is the distance from the axis of revolution to the point of intersection of the three surfaces. He also deduced certain meridian curves which have been discussed by Haentzschel in Reduction der Potentialgleichung, Berlin, 1893.

In reducing Laplace's equation the terms involving θ are removed in the usual manner, leaving a partial-differential equation which, as both writers state, is resolvable into two ordinary differential equations provided F and F_1, its conjugate, can be found which shall satisfy

$$\frac{F'(t+iu) \cdot F_1'(t-iu)}{[F(t+iu) - F_1(t-iu)]^2} = H(t) + K(u). \tag{2}$$

After F is found the two families of meridian curves follow from

$$x + ir = F(t+iu), \; x - ir = F_1(t-iu), \; r = \sqrt{(y^2 + z^2)}, \tag{3}$$

by elimination of u and t respectively, thus making t and u the parameters of which R_1 and R_2 are functions as stated above. As will be seen, H and K disappear in the process of finding F, and the principal result in this paper is a direct process of computing H and K, avoiding the method of substitution in the first member of (2), which must be followed by a separation not always obvious.

Wangerin differentiates (2) successively with respect to t and u respectively, obtaining

$$
\begin{aligned}
(4)\quad & (F - F_1)[(F - F_1)(F'''F_1' - F'F_1''') \\
& + (F''F_1' - F'F_1'')(F' - F_1') \\
& - 2F'F_1'(F'' + F_1'') - 2(F' + F_1')(F'F_1'' + F''F_1')] \\
& - 3(F' - F_1')[(F - F_1)(F''F_1' - F'F_1'') \\
& - 2F'F_1'(F' + F_1')] = 0.
\end{aligned}
$$

This expression is next differentiated three times with respect to the argument of F and then the derivatives of F_1 are eliminated. From this result comes the final quotation from Wangerin, the following equation *defining* F:

$$(5)\qquad \text{“}(F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).\text{”}$$

Corresponding to (5)

$$(6)\qquad (F_1')^2 = A_1F_1^4 + 4B_1F_1^3 + 6C_1F_1^2 + 4B_1'F_1' + A_1'.$$

The conjugate imaginary constants in (5) and (6) come from integration but must be taken *real,* because (4) is satisfied by by F, F_1 and their derivatives from (5) and (6) when and only when the following conditions hold:

$$(7)\quad A - A_1 = B - B_1 = C - C_1 = B' - B_1' = A' - A_1' = 0.$$

Haentzschel gives as the solution of (5)

$$
\begin{aligned}
(8)\quad F(z) &= z_0 \\
&+ \frac{\sqrt{R(z_0)}\sqrt{S} + \frac{1}{2}R'(z_0)[s - \frac{1}{24}R''(z_0)] + \frac{1}{24}R(z_0)R'''(z_0)}{2[s - \frac{1}{24}R''(z_0)]^2 - A\cdot R(z_0)},
\end{aligned}
$$

$$
\begin{aligned}
& S = 4s^3 - g_2s - g_3 = 4(s - \epsilon_1)(s - \epsilon_2)(s - \epsilon_3), \\
(9)\qquad & s = \wp(z), \qquad g_2 = AA' - 4BB' + 3C^2, \\
& \qquad\qquad\qquad g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3.
\end{aligned}
$$

The constant z_0 is in general unrestricted. When z_0 is a root of $R(z) = 0$, (8) reduces to

$$F(z) = z_0 + \frac{\frac{1}{4}R'(z_0)}{s - \frac{1}{24}R''(z_0)}. \tag{10}$$

Formula (8) was published in 1865 in a pamphlet entitled "Problemata quaedam mechanica functionum ellipticarum ope soluta.—Dissertatio inauguralis," by G. G. A. Biermann (Berolini), where it is quoted as derived from Weierstrass's lectures.

For $F(t + iu)$ given by (10) and also for

$$F(t + iu) = \wp(t + iu),$$

which is obtainable from (10) by linear transformation, Haentzschel has found (l. c.) the values of H and K in (2) by substitution of the respective values of F and F_1 in the first member. This method evidently requires a considerable knowledge of the properties of F and F_1 in order that the separation of the first member into two functions may be effected, thus obtaining the values of H and K as desired, and an inspection of Haentzschel's presentation will confirm this inference.

This paper gives a method of expressing H and K as functions of F and F_1 and requires only known properties of $\wp(z)$ in addition to the differential equation (5), thus giving explicitly the separation implied in (2). From (2) by differentiation as to t

$$\frac{(F - F_1)(F'F_1'' + F_1'F'') - 2F'F_1'(F' - F_1')}{(F - F_1)^3} = H'(t), \tag{11}$$

in which the arguments of F and F_1 are respectively $t + iu$ and $t - iu$, and it should be noted that F and F_1 are known functions previously determined by (5) and (6) and given by (8).

Although (11) appears to contain functions of u in the first member, the second member shows that it is independent of u and thus true for any value of u, e. g., $(t - c)/i$. This value of u was suggested by the fact that $F_1(t - iu)$ thus becomes $F_1(c)$. If next c is so chosen as to satisfy

$$F_1'(c) = 0, \tag{12}$$

equation (11) will be much simplified. Since F given by (8) is a function of $\wp$, (12) is satisfied by taking c as ω, the half period of $\wp$, noticing that $\wp'(\omega)$ is 0. Hence (11) becomes

$$\frac{F_1''(\omega)F'(2t-\omega)}{[F(2t-\omega)-F_1(\omega)]^2}=H'(t). \tag{13}$$

Integrating with respect to t gives

$$H(t)=\frac{-F_1''(\omega)}{2[F(2t-\omega)-F_1(\omega)]}+C_1. \tag{14}$$

Similarly

$$K(u)=\frac{F_1''(\omega)}{2[F(2iu+\omega)-F_1(\omega)]}+C_2. \tag{15}$$

The final form of (2) is

$$\frac{F'(t+iu)\cdot F_1'(t-iu)}{[F(t+iu)-F_1(t-iu)]^2}=H(t)+K(u) \tag{16}$$
$$=\frac{F_1''(\omega)}{2}\left[\frac{1}{F(2iu+\omega)-F_1(\omega)}-\frac{1}{F(2t-\omega)-F_1(\omega)}\right].$$

The term C_1+C_2 from the previous equations is omitted as (16) is an identity for $iu=t-\omega$.

Before treating the general value of F by the use of (16) several special cases will be computed. For the simplest case, $F=\wp$, obtainable from (10) by a linear transformation with real coefficients, the following are necessary in applying (16):

$$\wp(\omega)=e_1,\ \wp'(\omega)=0,\ \wp''(\omega)=2(e_2-e_1)(e_3-e_1),$$

$$\wp(2t-\omega)-e_1=\frac{(e_2-e_1)(e_3-e_1)}{\wp(2t)-e_1}, \tag{17}$$

$$\wp(2iu+\omega)-e_1=\frac{(e_2-e_1)(e_3-e_1)}{\wp(2iu)-e_1},$$

whence

$$H(t)+K(u)=\wp(2iu)-\wp(2t). \tag{18}$$

When F is given by (10) and z_0 is a real root of $R(z)=0$, the result is (18) as in the preceding case, the computation being somewhat longer, though depending only on (17). In this case the coefficients in (10) are real, so that either of the values of F in the two cases is a real linear transformation of the other. It is a remarkable fact that under such a transformation the fundamental formula (16) is invariant, this being true also of the more general form which (16) would have if c were not specialized.

To verify the proposition above for the last member of (16), let $F(2t - \omega)$ be replaced by $(\alpha F + \beta)/(\gamma F + \delta)$, $\alpha, \beta, \gamma, \delta$ real, and $\alpha\delta - \beta\gamma = \Delta \neq 0$, and similarly for $F(2iu + \omega)$, writing also a like function of W in place of $F_1(\omega)$, noticing that W' will be zero. The result is

$$(19)\quad \frac{W''\Delta}{2(\gamma W+\delta)^2}\left[\frac{(\gamma F_1+\delta)(\gamma W+\delta)}{(F_1-W)\Delta}-\frac{(\gamma F+\delta)(\gamma W+\delta)}{(F-W)\Delta}\right]$$

$$= \frac{W''}{2}\left[\frac{1}{F_1 - W} - \frac{1}{F - W}\right].$$

When at least two of the roots of $R(z) = 0$ are imaginary let z_0 be one of the conjugate pair a_1 and a_4; then from (10)

$$(20)\quad F(t + iu) = a_1 + \frac{\frac{1}{4}R'(a_1)}{\wp(t - iu) - \dfrac{R''(a_1)}{24}}$$

$$= a_1 + \frac{M_1}{\wp(t + iu) - N_1},$$

$$F_1(t - iu) = a_4 + \frac{\frac{1}{4}R'(a_4)}{\wp(t - iu) - \dfrac{R''(a_4)}{24}}$$

$$= a_4 + \frac{M_4}{\wp(t - iu) - N_4},$$

$$F_1(\omega) = a_4 + M_4/(e_1 - N_4),\quad F_1'(\omega) = 0,$$

$$F_1''(\omega) = - M_4\wp''(\omega)/(e_1 - N_4)^2.$$

These values substituted in (16) give

$$(21)\quad H(t) + K(u) =$$

$$\frac{M_4\wp''(\omega)}{2(e_1 - N_4)^2}\left[\frac{1}{a_1 - a_4 + \dfrac{M_1}{\wp(2t - \omega) - N_1} - \dfrac{M_4}{e_1 - N_4}}\right.$$

$$-\frac{1}{a_1 - a_4 + \dfrac{M_1}{\wp(2iu + \omega) - N_1} - \dfrac{M_4}{\epsilon_1 - N_4}}\Bigg].$$

For the reduction of (21) the following formulas are useful:

$$
\begin{aligned}
(22)\quad R(z) &= A(z - a_1)(z - a_2)(z - a_3)(z - a_4),\\
R'(a_1) &= A(a_1 - a_2)(a_1 - a_3)(a_1 - a_4) = 4M_1,\\
R''(a_1) &= 2A[(a_1 - a_2)(a_1 - a_3) + (a_1 - a_3)(a_1 - a_4)\\
&\qquad + (a_1 - a_4)(a_1 - a_2)] = 24N_1
\end{aligned}
$$

[similarly for M_4 and N_4],

$$\epsilon_1 = \frac{A}{12}[(a_1 - a_2)(a_3 - a_4) + (a_1 - a_3)(a_2 - a_4)],$$

$$\epsilon_2 = \frac{A}{12}[(a_2 - a_3)(a_1 - a_4) + (a_2 - a_1)(a_3 - a_4)],$$

$$\epsilon_3 = \frac{A}{12}[(a_3 - a_1)(a_2 - a_4) + (a_3 - a_2)(a_1 - a_4)],$$

$$\epsilon_1 - N_4 = \frac{A}{4}(a_3 - a_4)(a_4 - a_2),$$

$$\epsilon_1 - N_1 = \frac{A}{4}(a_1 - a_2)(a_3 - a_1),$$

$$a_1 - a_4 - \frac{M_4}{\epsilon_1 - N_4} = 0.$$

With the aid of (17) also, the result for this case is

$$(23)\quad H(t) + K(u) = -(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3)\left[\frac{1}{\wp(2t) - \epsilon_1} - \frac{1}{\wp(2iu) - \epsilon_1}\right].$$

It will be noticed that ϵ_2 and ϵ_3, which are conjugate, enter symmetrically, ϵ_1 being real. When all of the roots of $R(z) = 0$ are imaginary and accordingly all of the ϵ's are real, the choice of z_0 as in the preceding case reproduces (23).

The final forms for each case up to this point are the same as given by Haentzschel, though the method is different, in that one formula, (16), gives the resolved form for all cases.

For the last and most general case, where F is given by (8), the complete resolution into separate functions of t and u respectively would otherwise be obtainable only with much difficulty.

Although (16), the principal formula in this paper, is complete theoretically, there are three ways in which simplifications may be made in the general case. The first depends upon changes in the form of F which the writer has given in the *Archiv der Mathematik und Physik*, III, volume 23, Heft 1, and volume 24, Heft 4. The second involves certain simplifications in $F_1(\omega)$ which, with other related matters, are intended for a later paper, while the third consists of the following substitutions derived from the present discussion. In the last member of (16), s in F in the second denominator is replaced by $\wp(2t - \omega)$, also

$$(24) \qquad S = \frac{4(\epsilon_1 - \epsilon_2)^2(\epsilon_1 - \epsilon_3)^2[\wp(2t) - \epsilon_2][\wp(2t) - \epsilon_3]}{[\wp(2t) - \epsilon_1]^3}$$

and similarly for the first denominator.

In forming $F_1(\omega)$, z_0 is replaced by its conjugate, but s is simply ϵ_1 which is real, while S is zero.

$F_1''(\omega)$ may be computed from (6) thus depending on $F_1(\omega)$, i. e.,

$$(25) \qquad F_1''(\omega) = 2[AF_1^3(\omega) + 3BF_1^2(\omega) + 3CF_1(\omega) + B'].$$

As a summary, it may be stated that Wangerin made the solution of the original potential problem depend upon expressing a certain function of an unknown function as a sum of two functions $H(t)$ and $K(u)$, and obtained some simple values of the unknown function. Haentzschel applied a formula due to Weierstrass thus finding a general value of the unknown function, and he also computed the values of the two functions H and K for special cases only. Formulas (16) and (25) above give H and K explicitly in terms of the general function mentioned.

UNIVERSITY OF PENNSYLVANIA,
August 1, 1916.

NOTE CONCERNING THE NUMBER OF POSSIBLE INTERPRETATIONS OF ANY SYSTEM OF POSTULATES.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society December 27, 1917.)

In this note it is assumed that a mathematical system of postulates contains one or more undefined terms and that at least one of these denotes an element, i. e., a thing as distinguished from a relation. The assignment of an admissible meaning, or value, to each of the undefined elements of a postulate system will be spoken of as an interpretation of the system. By "admissible" meanings are meant meanings that satisfy the postulates or that, in other words, render them true propositions.

A postulate system may be such that from a given interpretation of it a second interpretation may be derived, from the second a third one, from the third a fourth, and so on, in such a way that mathematical induction is available for proving that the system admits of a denumerable infinitude of different interpretations. Such a system, for example, is Hilbert's system for euclidean metric geometry, where the undefined element-names are point, line, and plane. It is well known that one interpretation of this system results from allowing the term point to mean an ordered triad (x, y, z) of real numbers, the term plane to mean the class of triads satisfying an equation of the form $Ax + By + Cz + D = 0$ where not all the coefficients vanish, and the term line to mean the class of triads satisfying a pair of such equations.

If now we let the equations

$$(1) \quad \pi_1 + \lambda\pi_2 = 0, \qquad \pi_3 + \mu\pi_4 = 0, \qquad \pi_5 + \nu\pi_6 = 0$$

represent three pencils of planes, where the term plane has the meaning above assigned, it is evident that we may get a second interpretation of the Hilbert system by agreeing that the term point shall mean a triad (λ, μ, ν) of planes selected as indicated from the given pencils, that by a plane shall be meant a congruence, $a\lambda + b\mu + c\nu + d = 0$, of such triads, and by a line the one-parameter system of triads common to two such congruences.

To obtain a third interpretation it is sufficient to regard the equations (1) as now representing three pencils of congruences of the foregoing variety and then to take for a point a triad of such congruences, for a plane a congruence of such triads of congruences, and for a line a one-parameter system of them as before.

The process exemplified in obtaining the third interpretation from the second obviously admits of endless repetition, yielding a denumerable infinitude of interpretations of which no two are identical.

The sequence of interpretations thus obtainable, far from including all possible interpretations of the system in question, excludes infinitely many possibilities. One of these is the familiar interpretation which results from agreeing that by a point shall be meant an ordinary point of ordinary inversion space a specified point O of which is treated as non-existent, that a plane shall signify a sphere that would contain O if O were not treated as non-existent, and that a line shall mean the (defective) circle common to two such defective spheres. Starting with this interpretation it would be easy to derive in the way above exemplified an endless sequence of interpretations differing from one another and from those of the foregoing sequence. It is not difficult to see that it would be possible to devise an endless variety of such sequences.

It is noteworthy that in case of any one of the interpretations belonging to either of the two sequences above indicated the ensemble of things denoted by the term plane and the ensemble denoted by the term line are subclasses of the ensemble denoted by the term point; and it is sometimes assumed that this relation of the three terms is required by the Hilbert postulates. That the relation is not required may be shown by an example as follows. In this interpretation the term point is to mean a triad (λ, μ, ν) of lines selected one from each of the pencils

$$(2)\quad L_1 + \lambda L_2 = 0, \qquad L_3 + \mu L_4 = 0, \qquad L_5 + \nu L_6 = 0,$$

which may be in a same plane or in three different planes; the term plane is to signify a congruence $a\lambda + b\mu + c\nu + d = 0$ of conics selected from the conic pencils

$$(3)\quad C_1 + \lambda C_2 = 0, \qquad C_3 + \mu C_4 = 0, \qquad C_5 + \nu C_6 = 0,$$

taken to be in a same plane or in different planes; and the term line is to denote the one-parameter system of quadric surfaces common to a pair of congruences

$$(4)\quad \alpha\lambda + \beta\mu + \gamma\nu + \delta = 0, \qquad \alpha'\lambda + \beta'\mu + \gamma'\nu + \delta' = 0$$

of quadric surfaces selected from, say, three pencils of quadrics

$$(5)\quad Q_1 + \lambda Q_2 = 0, \qquad Q_3 + \mu Q_4 = 0, \qquad Q_5 + \nu Q_6 = 0.$$

The questions arise: Does every postulate system admit of an infinite number of interpretations? If so, is the number the same for all systems? If it is, what is the number?

Let S be any postulate system. By hypothesis it contains at least one undefined element-name. Denote it by e. Let I be an interpretation of S. Suppose the meaning assigned in I to e is e'. Then all instances, e_1', e_2', e_3', $\cdots$, of e' satisfy the postulates of S. Let O be any object of thought whatever. This object and any given instance of e' together constitute a pair. Let ϵ be the class of such pairs. Evidently a one-one correspondence subsists between the instances of e' and those of ϵ (the members of the class ϵ). If two or more instances, e_1', e_2', $\cdots$, of e' satisfy some relation, we may say that the corresponding instances, ϵ_1, ϵ_2, $\cdots$, of ϵ also satisfy the same relation, provided we agree to say, as we evidently may, that any instances of ϵ shall be regarded as satisfying a given relation whenever the e'-instances contained in the ϵ-instances satisfy the relation. Accordingly, if we assign to e the meaning ϵ instead of the meaning e', we thereby give S a new interpretation I'. As many different interpretations may be thus obtained as there are objects O. In general the difference between two such interpretations will be trivial but it will never be nothing. The conclusion is that any postulate system admits of any given infinite number of interpretations.

COLUMBIA UNIVERSITY,
December, 1917.

DARBOUX'S PRINCIPLES OF GEOMETRY.

Principes de Géométrie Analytique. By GASTON DARBOUX. Paris, Gauthier-Villars, 1917.

A MONTH before his death Darboux wrote the preface to this, his last treatise. It presents in coordinated form lectures which he had delivered at intervals since 1872 either at the Sorbonne or the Ecole Normale. In the preface he announces that "the essential object of the book is the precise development of the notions relative to the imaginary and to infinity, and to show that in geometry they have the same place and importance as has been attributed to them in analysis for a long time." This idea is one which had very strong hold on Darboux's mind during his whole life. We find evidences of it in his first treatise, Sur une Classe remarquable de Courbes et de Surfaces algébriques et sur la Théorie des Imaginaires, published in 1872, which deals largely with material presented in better form in part of the present volume. He touched upon the idea in his address before the Fourth International Congress at Rome in 1908. Moreover, the results of his own researches, in which isotropic elements are so effectively used, bear fitting testimony to the force of his contention.

The treatise consists of five books, or parts. The first has an introductory chapter dealing briefly with the history of the development of the ideas of infinity and imaginary elements, and an indication of the value of these entities and of homogeneous coordinates. Darboux remarks that the moderns differ from the ancients in their aim to reduce geometrical problems to general principles. His own efforts along this line in the field of differential geometry were rewarded with results which constitute one of his chief glories. The next three chapters deal respectively with tetrahedral coordinates, homographic point transformations, anharmonic ratio, abridged notation of Bobillier and plane homologies. The treatment is essentially the same as is to be found in other developments of these ideas from the analytic point of view. It is necessary to treat these ideas fully, since the treatise is intended for students who have had merely an introductory course in cartesian analytical geometry of two dimensions. Similar characterization may be given the next three chapters, which are short, and which deal with biaxial homologies, the principle of duality and correlation respectively. The last chapter of the first book deals with the well known projective properties of conics treated analytically.

In the second book isotropic elements are defined analytically and used in the development and refinement of the conceptions of distance and angle. The first chapter deals with metrical relations in the plane. The formula of Laguerre for the angle α between two lines D and D', namely, that $e^{2i\alpha}$ is equal to the cross ratio of the lines D, D' and the isotropic directions of the plane, is derived, and applied to the derivation of the formulas of the trigonometry of triangles, whether real or imaginary. The isotropic lines through two points A, B in a plane meet in two other finite points A', B' called by Darboux associate to the first. If M is any other point of the plane, we have

$$\frac{AM}{BM} = (-1)^k e^{i\widehat{B'MA'}},$$

k being an integer By means of this formula Darboux shows that in a double infinity of ways the circle is the locus of a point whose distances from two fixed points, or poles, are in constant ratio. Generalizing this result, he studies during the remainder of the chapter the curves satisfying the condition $R_1R_2\cdots R_n = kr_1r_2\cdots r_p$, where k is constant, R_i is the distance of any point of the curve to a fixed point A_i and r_j the distance to a point a_j.

In chapter two the methods and ideas previously developed are applied to the determination and study of two classes of curves, obtained as solutions of the two problems:

To find the curve whose tangent at any point intercepts on n fixed lines of the plane segments which, measured from any fixed points on the lines, are in the most general linear relation.

To find the curve enveloped by a line which forms with n couples of fixed lines n triangles whose perimeters are in linear relation.

The imaginary circle at infinity, the isotropic plane, and null sphere, or isotropic cone, are defined analytically in chapter three; the idea of distance is made precise, and the angle between two lines is expressed in terms of cross ratio.

One of the best examples of the simplicity introduced into analytical geometry by the use of isotropic elements is furnished by taking the rectilinear generators of the sphere for its parametric lines. These parameters are introduced in the fourth chapter, and the relation between rotations of the sphere and the linear fractional transformation is established. Having proved that the cross ratio of any four points on a circle of the sphere is equal to the cross ratio of the four generators of the one system through these points, and also of the four generators of the second system through the points, Darboux establishes the following fundamental formula

$$\frac{\tan \cdot \frac{1}{2} AM}{\tan \frac{1}{2} AM'} = e^{i\overline{\gamma\gamma'}},$$

where AM and AM' are two arcs of great circles, and γ and γ' are poles of these circles, suitably chosen so that conventions as to signs shall be maintained. This formula forms the basis for the derivation, in the next chapter, of the formulas for the trigonometry of the sphere, such that the sides of the triangle are not required to be real, nor in the real case restricted in magnitude.

If A and B are two points of the sphere not on the same generator, the two generators through A meet the two generators through B in two points A' and B' other than points on the circle at infinity. The segments AB and $A'B'$, which evidently are polar to one another with respect to the sphere, are called associate segments. If a and b are points diametrically opposite to A and B, the four generators of the first system d, d_1, d_2, d_3 through these points meet the four generators of the second system $\delta, \delta_1, \delta_2, \delta_3$ in twelve points at finite distance. We indicate them as follows:

$$\begin{array}{cccccccccccc} A & B & A' & B' & A'' & B'' & a & b & a' & b' & a'' & b'' \\ d\delta_1 & d_2\delta_3 & d_2\delta_1 & d\delta_3 & d\delta_2 & d_1\delta_3 & d_1\delta & d_3\delta_2 & d_1\delta_2 & d_3\delta & d_2\delta & d_3\delta_1. \end{array}$$

In the last chapter of the second book, Darboux makes use of this enlarged system of points to study analytically the geometry on the sphere. He establishes the following fundamental system of formulas in which γ_A denotes the distance of any point M on the sphere to the point A, and S_{ABM} the area of the triangle ABM:

$$\frac{\gamma_A \gamma_B}{\gamma_{A'} \gamma_{B'}} = \frac{\gamma_a \gamma_b}{\gamma_{a'} \gamma_{b'}} = \cos \frac{AB}{2}, \quad e^{i\widehat{AMB}} = \frac{\gamma_{A'} \gamma_{b'}}{\gamma_{B'} \gamma_{a'}},$$

$$e^{\iota/2}S_{AMB} = \frac{\gamma_{b'}}{\gamma_{a'}}, \quad e^{\iota/2}S_{a'b'M} = \frac{\gamma_A}{\gamma_{B'}},$$

$$e^{\iota(\widehat{BAM}+\widehat{MBA})} = \frac{\gamma_{B'}\gamma_{b'}}{\gamma_{A'}\gamma_{a'}} = \frac{\sin B'M}{\sin A'M}.$$

These equations are applied to the determination and study of certain curves on the sphere analogous to those for the plane previously referred to.

Book Three is devoted to the proof of the celebrated theorems of Poncelet concerning polygons inscribed in one conic and circumscribed to a second conic. The analysis is rendered simple and elegant by the use of a coordinate system in the plane suggested by the Chasles representation of a quadric on a double plane. Each point of the plane is determined by the intersection of two tangents to a basal conic K. By a convenient choice of the equation of K the equations involved are given a simple form. After developing this coordinate system, the following theorems of consequence in the subsequent study are established:

A. If n lines are defined by equations of the form $A_i = 0$ $(i = 1, \cdots, n)$, the general equation of curves of order $n - 1$ passing through all the common points of the lines is

$$\frac{a_1}{A_1} + \cdots + \frac{a_n}{A_n} = 0,$$

the constants a_i being arbitrary.

B. If a curve of order $n - 1$ contains all the points of intersection of n tangents to a conic, it contains also the intersections of an infinity of other systems of n tangents to the same conic, each system being determined by a value of a certain parameter.

These theorems are applied at the beginning of the second chapter to the proof of Poncelet's theorem: When a polygon is circumscribed to a conic K and inscribed in a conic C, there exists a continuous suite of polygons of the same number of sides satisfying the same conditions. Moreover, in each case the points of intersection of any two sides of the polygon describe a conic or a straight line. After obtaining further properties of Poncelet polygons, Darboux gives an analytical explanation of the impossibility of obtaining such polygons when the conics C and K are chosen arbitrarily.

When a polygon of n sides is circumscribed to a conic K and $n - 1$ of its vertices, in a displacement, describe conics in the same tangential pencil* with K, the other vertex describes a conic of the pencil. This general theorem of Poncelet is the subject of the last chapter of the third book. Darboux gives a geometrical proof of the case $n = 3$, and then extends the proof to the general case. Afterwards an analytical proof of the theorem is given. The chapter closes with a discussion of the forms given by Chasles to this theorem.

In his early writings Darboux made use of the ideas of distance and angle which grew out of Cayley's work. In 1895 he developed in a course of lectures at the Sorbonne the theory of Cayleyan geometry. In revised form these lectures constitute Book Four of the present treatise. In the first chapter we find an analysis of the origin of Cayleyan geometry. There is a full geometrical and analytical discussion of the absolute and of the consequences of its use.

Displacements in Cayleyan space are those homographic transformations which leave invariant the absolute, and thus preserve distances and angles. Darboux shows, in the second chapter, that in order to obtain a continuous transformation of this kind it is necessary and sufficient to subject the parameters of the two families of rectilinear generators of the absolute to any two linear substitutions.

When one of these substitutions is the identity, in the corresponding displacement all points of space are displaced the same distance. For the general case, in every displacement there are two lines, D and Δ, polar with respect to the absolute, which slide over themselves; they are called the axes of the displacement and are real for a real displacement. These displacements are of three types, all of which may be real: helicoidal, when four points only of the absolute remain fixed; rotations, when all the points of either axis remain fixed; half-turns (retournements), when all the points of both axes are fixed. These results are derived geometrically, and then the equations of the displacements of various kinds are obtained in simple form, when the equation of the absolute is $xy + zt = 0$.

In the first part of chapter three the formulas of Cayleyan trigonometry are derived. Then follow expressions for volume and area. A sphere being defined as the locus of points equidistant from a fixed point, two cases present themselves according as the latter point does or does not lie on the absolute.

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* All the conics tangent to four lines form a tangential pencil.

The greater part of this book deals with finite elements, but occasionally there are sections dealing with differential geometry. The exposition throughout is so clear and simple that a college student might very well use this treatise as an introduction to this field of study.

The last book deals with inversion and with geometric concepts in whose field this transformation plays an important rôle. As a student in Paris, Darboux became interested in this subject and his first papers dealt with curves and surfaces for which these transformations are of peculiar significance. Out of them grew his remarkable thesis, Sur les surfaces orthogonales, and his first treatise previously referred to. In these publications are to be found the essential elements of the present book, so that here we have Darboux renewing his youth and embellishing its products with the style and finish acquired through the years.

Darboux begins with the following geometrical definition of inversion: Consider a sphere S, called the principal sphere of the inversion; the transform of a point M is the unique point M' such that the null spheres with centers at M and M' meet S in the same circle. From this he derives the well known equations of inversion. The first chapter is devoted to the determination of the essential properties of inversion. It should be noted that these are obtained by geometrical reasoning and not by analytical processes. In particular, isotropic lines are transformed into isotropic lines. Here is introduced the isotropic developable, that is, the developable whose generators are isotropic lines. These surfaces, which are transformed into surfaces of the same kind by inversion, play an important rôle in many of Darboux's writings. In one of his first papers he called attention to the fact that an isotropic developable circumscribing a surface meets it along a line of curvature. A double curve on the isotropic developable circumscribing a surface is the locus of null spheres doubly tangent to the surface. These are called the focal curves of the surface, and each point is a focus of the surface. This definition is a generalization of Plücker's for foci of a plane curve and is in accord with Chasles's definition of focals for quadrics.

If we have five fixed spheres such that any two meet one another orthogonally, and s_i denotes the power of a point with respect to the sphere S_i and r_i its radius, the following identities hold for any point:

$$(1)\qquad \sum_{i=1}^{5}\left(\frac{s_i}{r_i}\right)^2 = 0, \quad \sum_{i=1}^{5}\frac{s_i}{r_i^2} = -2, \quad \sum_{i=1}^{5}\frac{1}{r_i^2} = 0.$$

When one of the spheres S_i is a plane, these identities hold provided that in the first the term s_i/r_i be replaced by twice the distance from the point to the plane, and in the last two identities the corresponding term be omitted. The quantities x_i defined by

$$x_i = \lambda\frac{s_i}{r_i},$$

where λ is a factor of proportionality, are called the pentaspherical coordinates of the point. From the first of (1) we see that they must satisfy the condition

$$(2)\qquad \sum_{i=1}^{5} x_i^2 = 0.$$

Moreover, the factor λ is determined in accordance with the second of (1) by

$$\sum\frac{x_i}{r_i} = -2\lambda.$$

This coordinate system is introduced in the second chapter and various properties are obtained, including the relations between the pentaspherical and rectangular coordinates of a point. In Darboux's opinion these coordinates bear a peculiar relation to inversion just as homogeneous point coordinates do to homographic transformations and tangential coordinates to correlations. The contents of this chapter are in the main equivalent to those set forth by Darboux in the first volume of his Leçons sur la Théorie générale des Surfaces with the difference that in the present treatise there are fewer applications to differential geometry. In the present treatment these coordinates appear in their historical setting. For Darboux was lead to invent this system as a result of his study of cyclides, i. e., the surfaces of the fourth degree with the circle at infinity as a double curve and the surfaces of the third degree passing through this circle. The remainder of the treatise deals with the study of these surfaces.

According to Moutard, a surface is anallagmatic when it is transformable into itself by inversion. He showed that the envelope of the spheres orthogonal to a fixed sphere S possesses this property when S is the principal sphere of the inversion and that all anallagmatic surfaces can be obtained in this way. He called the center of S the principal pole of the surface and the locus of the centers of the spheres the directrix surface. He announced the following properties of the cyclides:

Every cyclide is anallagmatic in five different ways; the line joining any two principal poles is perpendicular to the plane of the other three; any two of the five principal spheres cut orthogonally; the directrix surfaces are homofocal quadrics, which are central quadrics when the surface is of the fourth order and paraboloids when of the third order; each sphere doubly tangent to the surface meets it in two circles; each principal pole of a cyclide of the fourth order is the vertex of a quadric cone whose generators are doubly tangent to the surface, the curve of contact being on a sphere concentric with the directrix quadrics, and through each point of the surface pass in general ten circular sections.

These results are established by Darboux in the third chapter by analytical processes. The equation in cartesian coordinates of the general cyclide is of the form

$$\varphi_0(x^2 + y^2 + z^2)^2 + 2\varphi_1(x^2 + y^2 + z^2) + \varphi_2 = 0,$$

where φ_0, φ_1, φ_2 denote functions of x, y and z of the degrees indicated by the subscripts. Darboux gives these functions normal forms before proceeding to the proof of the above result. He establishes also the existence of quadrics V inscribed in the cyclide of fourth order along curves lying on concentric spheres; the five cones referred to above are quadrics. Since a cyclide of the third order can be regarded in an infinity of ways as a homographic transform of a general cubic surface, certain properties of the latter are studied in relation to the generalized inversion proposed by Hirst.

We remarked that Darboux was led to pentaspherical coordinates by his study of the cyclides. When the five fixed spheres referred to in Moutard's results are taken for the coordinate spheres of a pentaspherical system, the equation of the general cyclide is of the form

$$\sum_{i=1}^{5} a_i x_i = 0,$$

as Darboux shows in chapter four. In this chapter and the next he derives from this analytical point of view the properties of the cyclides mentioned above, and many others.

Darboux's first important discovery was the existence of triply orthogonal systems of homofocal cyclides which constitute the most direct generalization of the well known triple system of confocal quadrics. The equation of such a system in pentaspherical coordinates is

$$\sum_{i=1}^{5} \frac{x_i^2}{a_i - \lambda} = 0,$$

where the a's are constants, and λ is a parameter, such that through each point of space pass three cyclides corresponding to three values of λ. Chapter six is devoted to a study of these systems.

The final chapter of the treatise deals with a point transformation first made known by the author in 1865. This transformation is defined analytically by the equations

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{2a}{x^2 + y^2 + z^2 + a},$$

where a is a positive or negative constant. A combination of this transformation and the general homographic one gives the most general transformation which makes a plane correspond to a sphere. To every point $M(X, Y, Z)$ there correspond two points m and m' of coordinates x, y, z and x', y', z', which are the centers of null spheres meeting the sphere S, defined by

$$x^2 + y^2 + z^2 - a = 0,$$

in the circle of intersection of S and the polar plane of M with respect to S. We will not enumerate the various results obtained by Darboux, but will content ourselves with his observation that this transformation permits one to substitute for Cayleyan geometry a new geometry in which circles orthogonal to a sphere play the rôle of straight lines.

The reader of this treatise is sure to be impressed by the striking use which Darboux has made of isotropic elements. By the use of them and of frequent geometrical reasoning he has been enabled to avoid tedious stretches of analytical work. It is the simplicity and elegance of the processes which impress one.

It seems almost superfluous to refer to the style of presentation of the results in this treatise. Every one in the least familiar with Darboux's writings is conscious of their lucidity and charm of style. And here we have the final product of this great expositor.

L. P. EISENHART.

SHORTER NOTICES.

On Growth and Form. By D'ARCY W. THOMPSON. Cambridge, University Press, 1917. 8vo. 16+793 pages. Price 21 shillings.

UNTIL his recent appointment to the chair of natural history at the University of St. Andrews, the author of this work was professor of natural history at University College, Dundee. He has long been prominent as a member of various fishery boards and conferences, serving, for example, as the British delegate to the Bering Sea fishery conference of 1897. He is the author of "A Glossary of Greek Birds," has edited Aristotle's Historia Animalium and various blue-books on North Sea investigations, and he has translated and edited Hermann Müller's Fertility of Flowers.

And now we have this elaborate volume containing the results of many years of observation and study in widely separated fields and constituting an "easy introduction to the study of organic form." The author tells us: "It is not the biologist with an inkling of mathematics, but the skilled and learned mathematician who must ultimately deal with such problems as are merely sketched and adumbrated here. I pretend to no mathematical skill, but I have made what use I could of what tools I had; I have dealt with simple cases, and the mathematical methods which I have introduced are of the easiest and simplest kind."

After the Introduction we have chapters with the following titles: 2. On magnitude; 3. The rate of growth (106 pages); 4. On the internal form and structure of the cell; 5. The forms of cells; 6. A note on absorption; 7–8. The form of tissues or cell-aggregates; 9. On concretions, spicules, and spicular skeletons; 10. A parenthetic note on geodetics; 11. The logarithmic spiral* (94 pages); 12. The spiral shells of the foraminifera; 13. The shapes of horns, and of teeth or tusks: with a note on torsion; 14. On leaf-arrangement or phyllotaxis; 15. On the shapes of eggs and of certain other hollow structures; 16. On form and mechanical efficiency; and 17. On the theory of transformations or the comparison of related forms.

The whole work is written with fluency, lightness of touch, and notably attractive phrasing fraught with a wealth of suggestion. Many historical notes and full bibliographies indicate the author's thorough familiarity with the literature of his subject. The mathematics involved is indeed elementary, although some of the discussion refers to results whose derivation must have required the use of the theory of least squares and of probabilities† (for example, in Chapter 3), while other parts involve such subjects as are summarized by O. Fischer in his Physiologische Mechanik‡ (for example in Chapters 2 and 16). In conclusion let us note a few of the vast number of special topics discussed in a most interesting manner.

Some of the illustrations of "hexagonal symmetry" in the seventh chapter are: (1) artificial cellular tissue formed by the diffusion in gelatine of drops of a solution of potassium ferrocyanide; (2) "hexagonal epithelium-cells of the pigment layer of the eye, external to the retina"; (3) optical sections of soap froth imprisoned between glass plates; and (4) the bee's cell. We are told that "the curious reader may consult Sir Thomas Browne's quaint and beautiful account in the Garden of Cyrus,§ of hexagonal (and also of quincuncial) symmetry in plants and animals which 'doth neatly declare how nature Geometrizeth, and observeth order in all things.'" In the seven-page account of the bee's cell Thompson sketches the correct history[*] of discussions of its form (by Pappus, Maraldi, Réaumur, Koenig, Maclaurin, Lhuilier, Buffon, P. Huber and Darwin), and the derivation, by calculus, of its minimum property.

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* In connection with this chapter it is noted that "a great number of spiral forms, both organic and artificial, are described and beautifully illustrated in Sir T. A. Cook's Curves of Life, 1914, and Spirals in Nature and Art, 1903."

† An excellent American work which gives a good idea of some kinds of mathematics employed in biology is C. B. Davenport's Statistical Methods with Special Reference to Biological Variation. Third edition. New York, 1914.

‡ Encyklopädie der mathematischen Wissenschaft, Band IV, 1, II, Heft 1, 1904. Not referred to by Thompson.

§ Published in 1658.

The mathematician will more than once take exception to statements made in the two paragraphs on pages 440–441 concerning "geodetic" curves on surfaces. These curves are introduced in connection with the discussion of spicules, helicoid formations as in the coil which stiffens the tracheal tubes of an insect (Chapter 10), etc.

Discussion of leaf-arrangement leads to the series of ratios: 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55, 55/89, 89/144, which are simply convergents of the continued fraction $\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots$, etc. Thompson refers, incorrectly, to this "celebrated series" as "commonly called the Fibonacci series." This designation would be more correctly applied to the series of numbers 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 (each term of which, after the second, is the sum of the two preceding) which comes up in the solution of a rabbit problem discussed by Leonardo of Pisa in his Liber Abbaci.[†] Fibonacci's series is sometimes called Lamé's series.[‡] It was well known to Kepler, who discussed it in connection with flower forms,[§] and it is closely connected with the division into golden section[||] and with a number of mathematical investigations by Binet, Lamé, Lucas, Catalan, Schlegel, and Landau.[¶]

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[*] Incorrect statements, from the time of Réaumur, have been made so often in this connection (e. g., Mach, Science of Mechanics, 1902, p. 453) that the corrections indicated by R. L. Ellis in his essay "On the Form of Bees' Cells" should be noted (Mathematical and Other Writings, Cambridge, 1863, p. 353).

[†] Scritti, tome 1, pp. 283–284.

[‡] Thompson makes a slip in connecting this series with "Father Bernard Lami, a contemporary of Newton."

[§] F. Ludwig, "Kepler über das Vorkommen der Fibonaccireihe im Pflanzenreich," *Bot. Centralbl.*, 1896, Bd. 68, p. 7.

[||] Two books on this subject are: (1) A. Zeising, Neue Lehren von den Proportionen des menschlichen Korpers aus einem bisher unbekannt gebliebenen und die ganze Natur und Kunst durchdringenden Grundgesetze. Leipzig, 1854; (2) F. X. Pfeifer, Der goldene Schnitt und dessen Erscheinungsformen in Mathematik, Natur und Kunst. Augsburg, 1885 —but Thompson regards the numbers of golden section as "devoid of biological significance" (p. 649).

[¶] Cf. *Amer. Math. Monthly*, May, 1918.

In *L'Intermédiaire des Mathématiciens* for 1895* Monsieur Henri Brocard suggested a study of the "fruit de l'hélianthe"—a comparison of the arrangement of the florets of the sunflower with such curves as spirals, epicycloids and the involute of a circle. In 1909 a remarkable sunflower photograph taken by M. Brocard was published along with notes: (1) by J. W. N. Le Heux, who found "lignes rapellant tout à fait la spirale d'Archimède (ou la spirale logarithmique), ou même d'autres spirales, selon la loi de variation du rayon vecteur," and (2) by E. Maillet who wrote as follows:

"Je ne puis m'empêcher d'être frappé de l'analogie de disposition que présentent les spirales de la photographie de M. Brocard avec celles qu'a rencontrées M. le commandant Hartmann, dans ses études sur la déformation des métaux, pour le cas d'un cylindre creux soumis à une pression intérieure, ou extérieure, ou pour le cas d'une plaque poinçonnée en son centre [*Revue d'Artillerie*, novembre (page 99–100) et décembre 1894, 1895 et 1896].

"M. Mesnager a montré (*Revue d'Artillerie*, mars 1898) qu'on pouvait, au moins en partie, expliquer la production de ces lignes en admettant que, dans la déformation d'un corp donné soumis à des efforts: 1° le glissement tend à se produire en un point le long de l'élément plan suivant lequel la valeur absolue du rapport R de l'effort tangentiel à l'effort normal (cohésion comprise) est maximum; 2° ce glissement, qui constitue *une déformation permanente*, se produit effectivement dès que R atteint une valeur constante f (analogue à un coefficient de frottement) caractéristique du métal.

"Voir encore Appell, *Bull. Soc. Math.*, tome XXVIII, 1900, p. 66.

"Je ne suis pas assez botaniste pour dire si des phénomènes de déformation tout à fait analogues au point de vue de la Mécanique peuvent se produire dans le développement de la fleur et du fruit de l'hélianthe (ou d'autres plantes) et si cela peut influer sur la disposition de la mosaïque ou même la déterminer."

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* Tome 2, p. 134, question 524; reproposed 1902, tome 9, pp. 138–139. Notes: 1904, tome 11, p. 95; 1908, tome 15, pp. 173–174; 1909, tome 16, pp. 101–102.

In response to another question in *L'Intermédiaire* (1688, by Espanet, Dec. 1899, tome 6, p. 269), concerning surfaces of sea shells, more particularly those of spiral form, Brocard, Godefroy and Escott gave a number of references to the literature of the subject (1900, tome 7, p. 40; 1901, tome 8, pp. 167, 314; 1910, tome 17, p. 155), nearly all of which, and many more, seem to have been considered by Thompson. One notable work which he appears to have overlooked is Haton de la Goupillière's "Surfaces Nautiloides" which occupies almost the whole of the third volume of *Annaes scientificos da Academia polytechnica do Porto*, Coimbra, 1908.

According to the volume before us* it seems that there is no doubt about the logarithmic spiral arrangement in the sunflower, or indeed that we have systems intersecting isogonally. Reference is made to a careful and elaborate study by H. H. Church,† which has been adopted by T. A. Cook in his Curves of Life. But of Thompson's comment in this connection I shall not do more than quote a single sentence: "On the analogy of the hydrodynamic lines of force in certain vortex movements, and of similar lines of force in certain magnetic phenomena, Mr. Church proceeds to argue that the energies of life follow lines comparable to those of electric energy, and that the logarithmic spirals of the sunflower are, so to speak, lines of equipotential."

Professor Thompson's work is recommended as one of the most notable and most readable of scientific books appearing in the past decade.

R. C. ARCHIBALD.

Hvorledes Mathematiken I Tiden Fra Platon Til Euklid Blev Rationel Videnskab. By H. G. ZEUTHEN. Reprinted from the D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturvidensk. og Mathem. Afd., 8. Række, I. 5. Copenhagen, 1917. 183 pp.

ANY work by Professor Zeuthen on the history of mathematics, even though it be a reprint from the memoirs of an academy, deserves to be brought to the attention of other scholars than those who may chance to see the original publication. This is especially true when the memoirs of the academy are printed in a language not generally familiar to scholars and therefore are not as frequently consulted as those which appear in languages more nearly international.

In this particular case there is the more reason for calling attention to the memoir because a summary is given in the French language so that all scholars may have easy access to the argument and, with this as a guide and with a knowledge of German, may follow the more important details in the text itself.

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* Pp. 639–640.

† Relation of Phyllotaxis to Mechanical Laws, Oxford, 1901–1903; cf., *Ann. of Botany*, vol. 15 (1901), p. 481.

It is only a commonplace truism that the mission of the historian is quite as much the interpretation of facts as the mere writing of chronicles. It is not the facts of the present war but the interpretation of these facts and the relation of their influence upon the world that will tax the powers of the future historian. It is this feeling that will strike the reader of Professor Zeuthen's memoir. What has been done is not to call attention to newly discovered materials but to study again the significance of certain important contributions to Greek mathematics in one of the most remarkable periods of the early growth of the science. Probably only a few years intervened between the death of Plato and the birth of Euclid; certainly less than half a century elapsed from the period of the former's activity to that of the rise of the Alexandrian school; and yet it is in this period that Professor Zeuthen finds the recognition of mathematics as a logical science.

Chapter I considers the period in general, characterized as it is by the transition from an intuitive geometry to the demonstrative form in which it has come down to us.

Chapter II discusses those features which characterize a "science raisonnée" such as Euclid sought to include in his Elements. These features include the definition, the postulate, and those "common notions" which we call the axioms. The importance of all of these was fully understood by Plato, and the influence of this master inspired his pupils to undertake those investigations that finally took scientific form in the work of Euclid.

Chapter III sets forth the demands made by Plato upon mathematics in its quality of a "science raisonnée." The master seems not to have been particularly dissatisfied with the progress of plane geometry, possibly under his own influence; but he felt that stereometry was not yet upon a scientific basis.

Chapter IV discusses the rise of the analytic method. The positive facts of elementary geometry were already fairly well known, but it was apparently Plato who suggested the analytic method, with the inverse synthetic form of proof, and it was the development of this method that made possible a work like Euclid's.

Chapter V records the names of some of those who appreciated the reform instituted by Plato and carried it on to fruition. These names include not only those whose major interest was mathematics, but, it need hardly be said, the name of Aristotle whose contributions to the study of logic assisted in the efforts of the geometer to attain the ideals which Plato had set forth.

Chapter VI returns to the conception of geometry before the reform of Plato, or rather, to the pre-Platonic psychology of form. The problem is to set forth the primitive and intuitive images of people, the early apperception of similarity, of the relation of one form to another, and of the possibility of the displacement of figures without alteration.

Chapter VII enlarges upon the study of displacements as mentioned in the preceding chapter; for example, as shown in the earlier work of Pythagoras and in the much later work of the Indian scholar Bhāskara.

Chapter VIII brings the theory of elementary displacements to a climax in the Elements.

Chapter IX summarizes the growth of the idea of similarity, tracing it to its culmination in Euclid's work.

Chapter X is devoted to a consideration of the notion of angle. Professor Zeuthen shows that the general idea of angle, as distinct from the idea of perpendicularity, arose from the study of similar figures, which of course included congruence as a special case. The rise and use of the horn-shaped angle (keratoeides) is also mentioned.

Chapter XI considers the generalization of demonstrations, including the nature of incommensurables and the use of the idea of the infinitesimal. The influence of Zeno and Eudoxus upon Euclid's treatment of the incommensurable case and upon the work of Archimedes is shown. The influence of Aristotle upon Eudoxus and Euclid is compared with that of Dedekind upon Weierstrass.

Chapter XII relates to the generalization of propositions by Euclid to meet the demands imposed by Aristotle, and shows that his postponement of the geometric treatment of the quadratic was due to the desire to give the most general form possible.

Chapter XIII deals with the question of the ideality of geometric figures, a question by no means new when Plato called it to the attention of his pupils.

Chapter XIV discusses Euclid's treatment of stereometry, particularly with reference to the distinction between congruence and symmetry in a space of three dimensions.

Chapters XV and XVI treat of the Elements, the appreciation accorded to it as a scientific work during two thousand years, and its standing as a scientific work to-day. The influence of Euclid upon the work of Apollonius is set forth in terms of appreciation that it would be well for all teachers of mathematics to consider.

The work is of the scholarly character that one would expect from a man of Professor Zeuthen's exceptional attainments, and it is earnestly to be hoped that he will publish the entire memoir in the French language. Indeed, it should not be too much to hope that Professor Zeuthen may revise all of his historical works and publish them in a uniform French edition before he is compelled to relinquish his writing and to take the scholarly ease that he has so well earned.

DAVID EUGENE SMITH.

A Manual of Mathematics. By R. G. HUDSON and J. LIPKA. New York, Wiley, 1917. 132 pp. Price $1.00.

A Table of Integrals. By R. G. HUDSON and J. LIPKA. New York, Wiley, 1917. 25 pp. Price 15 cents.

THIS manual is taken from the mathematical section of the authors' manual for engineers. It contains the formulas of algebra, trigonometry, mensuration, plane and solid analytic geometry, differential and integral calculus, ordinary and partial differential equations, complex quantities and vectors. In addition to the usual tables—four-place—there is a table of conversion factors for the various scientific units. The formulas are systematically arranged and the section headings are printed in heavy type so that it is easy to locate any formula, a very desirable feature in such a book. The table of integrals is taken from the manual. The integrals are expressed in terms of the coefficients used in the integrand. While this makes the table more difficult for the printer than if substitutions and abbreviations had been made, it is much more convenient for the student.

THOS. E. MASON.

NOTES.

THE April number (volume 40, number 2) of the *American Journal of Mathematics* contains the following papers: "On the representation of functions in series of the form $\Sigma c_n g(x + n)$," by R. D. CARMICHAEL; "Transformations of planar nets," by L. P. EISENHART; "Orthogonal function sets arising from integral equations," by O. D. KELLOGG; "Complete systems of concomitants of the three-point and the four-point in elementary geometry," by C. H. RAWLINS, JR.; "Systems of pencils of lines in ordinary space," by A. L. MILLER; "Some contributions to the geometry of plane transformations," by TOBIAS DANTZIG; "Properties of a certain projectively defined two-parameter family of curves on a general surface," by PAULINE SPERRY.

AN announcement has just been issued that the *Bollettino di Bibliografia e Storia delle Scienze Mathematiche,* edited by Professor GINO LORIA, of Genoa, is about to begin a new series. This valuable journal will hereafter be published by the Libreria Scientifica D. Capozzi, Palermo, Via Croce dei Vespri, 8. The *Bollettino* has proved so useful to students of the bibliography and history of mathematics that the inauguration of a new series becomes a matter of interest to all mathematicians.

AT the meeting of the Edinburgh mathematical society held February 8, the following papers were read: By G. SMEAL, "The parameter method of iteration"; by Miss E. PAIRMAN, "A new form of the remainder in Newton's interpolation formula."

At the meeting of March 8, the following papers were read: By C. G. KNOTT, "Note on the quaternion theory of confocals"; by R. F. MUIRHEAD, (a) "A graphic method of determining the maximum bending moment for a rolling load on a uniform beam"; (b) "Geometrical notes on the accelerations of points of a rigid plane body moving in its own plane."

AT the meeting of the London mathematical society held February 14 the following papers were read: By A. C. DIXON, "Note on functional equations which are limiting forms of integral equations"; by D. M. Y. SOMMERVILLE, "The singularities of trochoidal curves"; by L. J. MORDELL, "A statement of Fermat."

AT the meeting of the National academy of sciences at Washington, April 22–24, Professor EDWARD KASNER presented a paper on "Conformal geometry." Professor G. D. BIRKHOFF was elected a member of the academy.

GINN and Company announce that the second volume of Veblen and Young's Projective geometry is now in press.

THE Cambridge University Press announces the early publication of a Theory of functions of a complex variable, by A. R. FORSYTH. Longmans, Green and Company, London, are publishing a Differential equations by H. BATEMAN.

ON pages 183–267 of the Publications of the members of the University of Chicago 1902–1916, recently issued by the University of Chicago Press, are listed published doctors' theses and the publications of members of the staffs in the departments of mathematics, astronomy and astrophysics, and physics.

THE following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 8–August 16).—By Professor JAMES MACLAY: Fundamental mathematical concepts, five hours; Higher algebra, five hours.—By Professor EDWARD KASNER: Modern geometry, five hours; Continuous groups, five hours.—By Professor W. B. FITE: Theory of functions of a real variable, five hours; Differential equations, five hours.

COLUMBIA UNIVERSITY (academic year 1918–1919).—By Professor T. S. FISKE: Theory of functions, four hours.—By Professor F. N. COLE: Algebra, four hours.—By Professor JAMES MACLAY: Differential geometry of surfaces, four hours, first half-year; Applications of the elliptic functions, three hours, second half-year.—By Professor D. E. SMITH: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor C. J. KEYSER: Philosophy of mathematics, four hours.—By Professor EDWARD KASNER: Ballistics, two hours, second half-year; Seminar in differential geometry, two hours.—By Professor W. B. FITE: Differential equations, three hours.—By Dr. C. A. FISCHER: Calculus of variations, three hours, second half-year.

The following university and college teachers of mathematics have recently entered the national military service:

Dr. W. L. CRUM, of Yale University, has joined the national army. Professor G. C. EVANS, of Rice Institute, has joined the signal corps and has gone to France. Professor W. E. MILNE, of Bowdoin College, has been made first lieutenant in the ordnance reserve corps. Dr. H. C. M. MORSE, of Harvard University, has joined the ambulance corps and is in France. Professor F. R. MOULTON, of the University of Chicago, has been commissioned major in the ordnance reserve corps. Professor A. H. NORTON, of Elmira College, and Professor L. P. SICELOFF, of Columbia University, are engaged in Y. M. C. A. work in France. Dr. J. F. RITT, of Columbia University, has joined the ordnance at the Aberdeen proving ground, Aberdeen, Md.

PROFESSOR E. W. BROWN, of Yale University, has been elected a member of the council of the American philosophical society.

AT Harvard University assistant professor J. L. COOLIDGE has been promoted to a full professorship of mathematics.

AT Wellesley College, Professor ROXANA H. VIVIAN, of the department of mathematics, has been made director of the department of hygiene and physical education. Professor CLARA E. SMITH is to exchange with Professor FLORENCE P. LEWIS, of Goucher College, for the coming academic year. Dr. MARY F. CURTIS, of Western Reserve University, has been appointed instructor in mathematics.

DR. W. G. SIMON has been appointed instructor in mathematics at Adelbert College, Western Reserve University.

PROFESSOR A. BENTELI, of the University of Bern, died November 10, 1917, at the age of seventy years.

PROFESSOR E. OTT, of the University of Bern, died November 17, 1917, at the age of seventy years.

PROFESSOR F. R. HELMERT, of the University of Berlin, died June 15, 1917, at the age of seventy-three years.

THE death is reported of Professor CYPARISSOS STÉPHANOS, of the University of Athens.

LEWIS B. CARLL, the blind author of the first American treatise on the calculus of variations (published in 1881), died March 22, 1918, at the age of seventy-four years.

PROFESSOR H. R. HIGLEY, of the Stevens Institute of Technology, died May 1, 1918, at the age of fifty-four years.

BOOK CATALOGUES:—Galloway and Porter, Cambridge, England, list number 91, 149 titles in mathematics and physics. —Henry Sotheran and Co., 140 Strand, London, catalogue 770, exact and applied science, 928 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ASKWITH (E. H.). A course of pure geometry containing a complete geometrical treatment of the properties of the conic sections. New edition revised and enlarged. Cambridge, University Press, 1917. Demy 8vo. 11 + 284 pp. 7s. 6d.

CAREY (F. S.). Infinitesimal calculus. Section 2. London, Longmans, Green and Company, 1917. 10s. 6d.

CISOTTI (U.). Lezioni di analisi matematica, svolte nel r. Istituto tecnico superiore di Milano. Pavia, tip. Cooperativa, 1918. 8vo. 16 + 674 pp. L. 30.00

CRAWLEY (E. S.) and EVANS (H. B.). Analytic geometry. Philadelphia, E. S. Crawley, 1918. 8vo. 14 + 239 pp.

EVANS (H. B.). See CRAWLEY (E. S.).

INTERNATIONAL catalogue of scientific literature. Fourteenth annual issue. A: Mathematics. London, Harrison, 1917. 8vo. 8 + 163 pp. Paper. 15s.

MICHEL (C.). Cours d'algèbre et d'analyse. Paris, Alcan, 1916. 8vo. 10 + 860 pp.

Russell (B.). Mysticism and logic, and other essays. New York, Longmans, Green and Company, 1918. 8vo. 8 + 234 pp. $2.50

Shaw (J. B.). Lectures on the philosophy of mathematics. Chicago and London, Open Court, 1918. 8 + 206 pp. $1.50

Stadler (G.). Etudes sur l'équation $\delta^p u/\delta x^p = a\Delta^m u = 0$. (Diss., Lund.) Lund, 1916. 4to. 6 + 89 pp.

II. ELEMENTARY MATHEMATICS.

Brenke (W. C.). Advanced algebra. New York, Century Company, 1917. 8vo. 8 + 196 pp. Cloth. $1.25

——. Elements of trigonometry with tables. New York, Century Company, 1917. 8vo. 6 + 122 + 40 pp. Cloth. $1.25

Lynde (L. E.). See Newton (F. E.).

Newton (F. E.) and Lynde (L. E.). Exercises in plane geometry with syllabus and definitions. Designed for the second reading of the subject. 3d edition. Andover, Mass., Andover Press, 1918. 86 pp. $0.75

Pagliero (G.). Come si usano i logaritmi per calcolare rapidamente: teoria e pratica; tavola dei logaritmi a quattro decimali. Torino, ditta G. B. Paravia, 1917. 8vo. 47 pp. L. 2.00

Thompson (J. B). The art of teaching arithmetic. New York, Longmans, Green and Company, 1917. 8vo. 8 + 295 pp. $1.35

Unwin (W. C.). Short logarithmic and other tables. 6th edition. London, Spon, 1918. 1s. 6d.

Zoretti (L.). Tables numériques usuelles à l'usage des ingénieurs, des étudiants des facultés, des élèves des lycées et des écoles primaires supèrieures, etc. Paris, Gauthier-Villars, 1917. 52 pp. Cartonné. Fr. 3.00

III. APPLIED MATHEMATICS.

American ephemeris and nautical almanac, 1919. Washington, Government Printing Office, 1917. 8vo

Attwood (E. L.). A text-book of laying off, or the geometry of shipbuilding. 2d edition. New York, Longmans, Green and Company, 1918. 8vo. 8 + 123 pp. $2.00

Bucherer (E.). Grundzuge der mathematischen Geographie. Basel, Krebs, 1917. 8vo 40 pp.

Danford (R. M.). See Moretti (O.).

Effemeridi astronomiche ad uso dei naviganti per l'anno 1918 (Istituto idrografico della r. marina). Genova, tip. Istituto idrografico della r. marina, 1917. 8vo. 6 + 191 pp. L. 2.00

Flamard (E.). Calcul des systèmes élastiques de la construction. (Encyclopédie industrielle.) Paris, Gauthier-Villars, 1918. 8vo. 6 + 200 pp. Fr. 12.00

Flammarion (C.). Annuaire astronomique et météorologique pour 1918. Paris, Flammarion, 1918. 364 pp. Fr. 3.00

JAEGER (F. M.). Lectures on the principle of symmetry and its applications in all natural sciences. Amsterdam, Uitgevers Maatschappy Elsevier, 1917. 8vo. 333 pp.

MEMENTO Oppermann, à l'usage des ingénieurs, architectes, agents voyers, conducteurs de travaux, mécaniciens, industriels, entrepreneurs. Paris, Béranger, 1918. 16mo. 268 pp. Fr. 6.00

MORETTI (O.). Notes on training; field artillery details. Prepared under the direction of R. M. Danford. New Haven, Yale University Press, 1917. 8vo. 12 + 275 pp.

RODET (J.). Notions d'acoustique. Les instruments de musique. Le télharmonium. Paris, Gauthier-Villars, 1917. 8vo. 4 + 96 pp. Fr. 3.50

STOHLER (H.). Mathematische Geographie und sphärische Trigonometrie. Basel, Basler Drug und Verlags-Anstalt, 1916. 8vo. 96 pp.

TSCHAPPAT (W. H.). Text-book of ordnance and gunnery. New York, Wiley, 1917. 8vo. 10 + 705 pp. Cloth. $6.50

VALBREUZE (R. DE). Notions sommaires d'électrotechnique. Paris, chez l'auteur, 12, rue Pelleport, 1918. 8vo. 180 pp. Fr. 6.00

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

The thirty-first regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 6. There were present eighteen persons, including the following twelve members of the Society:

Professor R. E. Allardice, Professor Harry Bateman, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Dr. Pauline Sperry, Professor R. M. Winger.

Professor W. A. Manning occupied the chair. At the close of the session attending members were entertained at luncheon, and in the afternoon an interesting demonstration of experiments on aeroplane propellers was given by Professors Moreno and Lesley, of Stanford University.

It was provisionally decided that the next meeting at Stanford University should be held on April 5, 1919.

The following papers were read at this meeting:

(1) Professor D. N. Lehmer: "On Jacobi's extension of the continued fraction algorithm" (preliminary report).

(2) Professor H. F. Blichfeldt: "A second principle in the geometry of numbers."

(3) Professor M. W. Haskell: "Continuous groups of quadratic transformations."

(4) Professor D. N. Lehmer: "Arithmetical theory of certain Hurwitzian continued fractions."

(5) Professor E. T. Bell: "Numerical functions of $[x]$."

(6) Professor E. T. Bell: "Fourier series for certain elliptic functions."

(7) Professor R. M. Winger: "On the satellite line of the cubic."

(8) Professor R. M. Winger: "Self-projective rational septimics."

Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. Professor Lehmer developed new connections between

the ordinary continued fraction and the extension made by Jacobi in 1868, with applications to the theory of numbers and to the theory of equations.

2. In a former paper (*Transactions of the American Mathematical Society,* 1914, pages 227–235) Professor Blichfeldt proved a geometrical theorem having a number of applications in the theory of numbers; for instance, the variables of a positive-definite quadratic form f in n variables and of determinant D can be given such integral values, not all zero, that the numerical value of f is not greater than a number which has $(n/\pi e)D^{1/n}$ for its asymptotic value. This limit being too high, the question arises what may the actual lowest limit be; i. e., there is need of a theorem giving a limit which is exceeded by at least *one* quadratic form. Minkowski has given such a theorem in geometrical form (Collected Works, volume 2, page 270) without proof, namely: an oval surface (like an ellipse) in space of n dimensions, having a center at the origin and a volume $\leqq 2(1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots)$, can be deformed by a homogeneous linear transformation which preserves volumes, into a surface which contains no lattice points except the origin. Professor Blichfeldt states a theorem which gives somewhat better results than that of Minkowski; thus, applied to ellipsoids, this theorem is that the volume that may be assumed approaches a number $\geqq 23$ for large values of n (instead of 2 by Minkowski's theorem).

3. Professor Haskell has examined the various types of continuous transformations of the plane which in homogeneous form are rational quadratic transformations. They are all six-parameter groups of mixed quadratic and linear transformations, with two-parameter subgroups which are purely quadratic. Besides those which are simple products of linear transformations of the separate variables, they may be classified as elliptic, hyperbolic, and parabolic—the elliptic type including as a special case the well known inversion group.

4. If A_m/B_m is the mth convergent of the Hurwitzian continued fraction

$$\overline{(a_1, a_2, a_3, \cdots, a_{k-1}, ma_k)} \qquad (m = 1, 2, 3, \cdots)$$

and A_m'/B_m' that of

$$(\overline{a_{k-1}, a_{k-2}, \cdots, a_2, a_1, -ma_k - 2M}), \qquad (m = 1, 2, 3, \cdots)$$

and A_m''/B_m'' the $(m + 1)$st convergent of

$$(x, a_{k-1}, a_{k-2}, \cdots, a_2, a_1, y, \overline{a_{k-1}, a_{k-2}, \cdots, a_2, a_1, -ma_k - 2M}), \qquad (m = 2, 3, \cdots)$$

where

$$M = \frac{B_{k-1} + A_{k-2}}{A_{k-1}}, \quad x = \frac{B_{k-1} - A_{k-2}}{A_{k-1}},$$

$$y = - A_k - 2\left(\frac{B_{k-1}^2 + B_{k-2}A_{k-1}}{A_{k-1}B_{k-1}}\right)$$

and where $a_1, a_2, a_3, \cdots, a_{k-1}$ are positive or negative integers or zero, and a_k, A_{k-1}, B_{k-1} are positive or negative integers not zero, then Professor Lehmer shows

$$\begin{aligned} A_{\rho k-1} &= (-1)^{\rho-1}A'_{\rho k-1}, \\ A_{\rho k} &= (-1)^{\rho}(A'_{\rho k} + MA'_{\rho k-1}), \\ B_{\rho k-1} &= (-1)^{\rho-1}A''_{\rho k-1}, \\ B_{\rho k} &= (-1)^{\rho}(A''_{\rho k} + MA''_{\rho k-1}). \end{aligned}$$

From these equations he develops the arithmetical theory of such continued fractions, and shows, among other things, that for n prime to a_k, A_{k-1}, B_{k-1}, we have always

$$\begin{aligned} A_{2nk-1} &\equiv B_{2nk} \equiv 0 \pmod{n}, \\ A_{2nk} &\equiv B_{2nk-1} \equiv (-1)^{kn-1} \pmod{n}. \end{aligned}$$

In particular the denominators of the convergents of orders $3n$, $3n - 2$, and $3n - 6$, and the numerator of that of order $3n - 3$ for the regular continued fraction which represents the base of Naperian logarithms are always divisible by n.

5. Simple methods are given by Professor Bell for finding relations between, and reducing, sums of numerical functions of $[x]$, the greatest positive integer contained in x. Such reductions are useful in the investigation of arithmetical mean values, etc. The paper appeared in part in the March number of the *Annals of Mathematics*.

6. In Professor Bell's second paper the series are for the squares and certain products of the complete set of eighteen doubly periodic theta quotients considered by Hermite in his memoir on Kronecker's class-number relations. The series have important arithmetical consequences. The paper has appeared in the *Messenger of Mathematics*.

7. While Salmon discusses the satellite line of the cubic at some length he fails to give its equation. The symbolical expression for this important covariant has been supplied by Morley. In the present paper Professor Winger derives the explicit equation of the satellite, both for the rational and general cubic, in canonical forms, and discusses associated loci. Several chain theorems are obtained and a generalization is made for the plane curve of order n.

8. Professor Winger's second paper, a preliminary report of which was made to the Society at a former meeting, is intended as a sequel to two papers of similar titles which have already appeared in the *American Journal of Mathematics*. The varieties of self-projective rational septimics in parametric form are exhibited and the more striking properties of certain of the curves are pointed out. According to the criterion of classification adopted, there are 26 distinct types, invariant under cyclic groups of order 2, 3, 4, 5, 6, and 7, dihedral groups of order 6, 10, and 14, and an infinite group.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE tenth regular meeting of the American Mathematical Society at Chicago, being also the forty-first regular meeting of the Chicago Section, was held on Friday and Saturday, April 12 and 13, at the University of Chicago. The various sessions were attended by about fifty persons, among whom were the following thirty-five members of the Society:

Professor R. P. Baker, Professor G. A. Bliss, Professor J. W. Bradshaw, Professor R. D. Carmichael, Professor A. R.

Crathorne, Mr. G. H. Cresse, Professor L. E. Dickson, Professor Arnold Dresden, Professor J. W. Glover, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor A. M. Kenyon, Professor S. Lefschetz, Professor A. C. Lunn, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor C. N. Moore, Professor E. H. Moore, Professor R. E. Moritz, Professor E. J. Moulton, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Dr. G. W. Smith, Mr. L. L. Steimley, Professor E. B. Van Vleck, Professor L. G. Weld, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. A. Young.

Forty-two persons joined in a dinner at the Quadrangle Club on Friday evening, which was followed by a number of short informal speeches, President Dickson presiding.

At the session of Friday forenoon, Professor Hedrick read a paper in memory of Professor E. W. Davis, whose death occurred on February 3, 1918. A motion to transmit a copy of Professor Hedrick's paper to the family of Professor Davis was made by Professor Slaught and concurred in by a rising vote of the meeting.

Friday afternoon was devoted to a symposium on "Divergent series and summability." The principal papers were presented by Professor R. D. Carmichael, who spoke on "General aspects of the theory of summable series," and by Professor C. N. Moore, whose subject was "Applications of the theory of summability to developments in orthogonal functions." These papers were followed by a general discussion by Professors E. H. Moore, Hedrick, Curtiss, Hildebrandt, Bliss, and others.

Professor Bliss, chairman of the section, presided at the meetings on Friday; Professor Weld presided on Saturday morning.

At the sessions of Friday forenoon and Saturday forenoon, the following papers were read:

(1) Professor C. N. Moore: "On the summability of the developments in Bessel's functions."

(2) Mr. M. G. Smith: "On the zeros of functions defined by linear homogeneous differential equations, containing a parameter."

(3) Professor C. H. Sisam: "On surfaces containing a system of cubics which do not constitute a pencil."

(4) Mr. E. P. Lane: "Conjugate systems with indeterminate axis curves."

(5) Mr. R. F. Borden: "On the Laplace-Poisson mixed equation."

(6) Professor R. D. Carmichael: "On a general class of integrals of the form $\int_0^\infty \varphi(t)g(x+t)/g(x)dt$."

(7) Professor S. Lefschetz: "Abelian manifolds."

(8) Dr. P. R. Rider: "On the f_1 function of the calculus of variations."

(9) Professor E. J. Wilczynski: "Some geometric aspects of the theory of functions."

(10) Mr. L. J. Rouse: "A contribution to the question of linear dependence in linear integral equations."

(11) Professor G. A. Miller: "Determinant groups."

(12) Professor G. A. Miller: "Group theory proof of two elementary theorems in number theory."

(13) Professor Louis Brand: "Flexural deflections and statically indeterminate beams."

(14) Professor Arnold Emch: "Proof of Pohlke's theorem and its generalization by affinity."

(15) Dr. A. R. Schweitzer: "On the iterative properties of an abstract group. Second paper."

(16) Dr. A. R. Schweitzer: "Concerning independence."

(17) Professor T. H. Hildebrandt: "On a generalization of a theorem of Toeplitz."

Mr. Smith and Mr. Borden were introduced by Professor Carmichael, Mr. Lane by Professor Wilczynski, and Mr. Rouse by Professor Hildebrandt. The papers of Dr. Rider, Mr. Rouse, Professor Miller, Professor Brand and Professor Emch were read by title.

Abstracts of the papers follow, the numbers corresponding to those in the list above.

1. The principal object of Professor Moore's paper is to extend certain results of a paper presented to the Society at the last summer meeting (see this Bulletin, volume 24, page 65). In the previous paper the summability $(C, k > \frac{1}{2})$ at the origin, and the uniform summability $(C, k > \frac{1}{2})$ in the neighborhood of the origin, of the Bessel's development of an arbitrary function satisfying certain conditions, were established. In the present paper it is further

shown that the development is summable $(C, \frac{1}{2})$ at the origin and uniformly summable $(C, \frac{1}{2})$ in its neighborhood, under the same conditions on the arbitrary function. It is also shown that this is all that can be established for functions of the type considered, as an example is given of such a function for which the development is non-summable $(C, 0 \leqq k < \frac{1}{2})$ at the origin.

2. Using a certain generalization of a theorem due to Professor R. D. Carmichael and published in the *Annals of Mathematics* for March, 1918, Mr. Smith has derived conditions under which one can be assured that any solution of a differential equation of general order n has at least a given number of zeros in a fixed (finite) interval. It is found that the theorems are most easily applied in the case of an equation of even order of the form

$$E(u) \equiv \frac{d^{2m}}{dx^{2m}}[e_{2m}(x, \lambda)u] + \lambda^2 \frac{d^{2m-2}}{dx^{2m-2}}[e_{2m-2}(x, \lambda)u] + \cdots + \lambda^{2m}e_0(x, \lambda)u = 0.$$

For the particular case of $2m = 4$, one of the theorems takes the following form: If the parameter λ in the equation $E(u) = 0$ is fixed and satisfies the relations $\lambda \geqq S$, $\lambda \geqq s\pi/(b - a)$, and if for every x in the interval $a \leqq x \leqq b$ we have

$$e_0(x, \lambda) \geqq 3e_2 - 21e_4 + (60e_4 - 6e_2)\tan^2\lambda\,(x - a + \pi/2\lambda),$$

the equality sign not holding throughout (a, b), then any solution $u(x, \lambda)$ of $E(u) = 0$ has at least one zero in each subinterval of (a, b) of length π/λ measuring from the point a, and hence has at least s zeros in (a, b).

3. Professor Sisam's paper classifies completely the types of algebraic surfaces which contain an algebraic system of ∞^1 cubics which do not constitute a pencil, and discusses some fundamental properties of these surfaces.

4. The differential equation of the axis curves of a given conjugate system on a surface, and the conditions for their indeterminateness, were given by Professor Wilczynski in a paper

which appeared in the *Transactions*, volume 16. Making use of these conditions, Mr. Lane considers the problem of finding those surfaces on which axis curves are indeterminate. He reduces the problem to the integration of two ordinary linear homogeneous differential equations of the third order, and obtains a parametric representation for the coordinates of an arbitrary point on the most general surface of this kind. When two cones with a common vertex are given, a surface of the kind under consideration is determined, except for a three-parameter group of projective transformations, which leave invariant the common vertex of the two cones and every straight line through this point. The fundamental conjugate system on the surface then consists of the two one-parameter families of plane curves cut out on the surface by the tangent planes of the two cones. A number of geometric theorems concerning certain closely allied curves and surfaces are obtained.

5. This paper by Mr. Borden deals with the elementary theory of the mixed equation

$$(1) \quad f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0$$

along lines initiated by Poisson (*Journal de l'Ecole Polytechnique*, tome 6 (1806), pages 127–141), and in intimate relation to the analogous theory of the Laplace equation

$$s + ap + bq + cz = 0$$

as developed for instance in Forsyth's "Theory of Differential Equations," volume 6, pages 44–96.

The mixed equation has two fundamental invariants under the group of transformations

$$f(x) = v(x)g(x).$$

If these invariants are properly chosen, the equation can be solved in finite form if one of them is zero. The equation is unchanged in form by the Laplace-Poisson transformations

$$(S) \qquad f_{S_1}(x) = f(x+1) + p(x)f(x),$$

$$(T) \qquad f_{T_1}(x) = f'(x) + q(x-1)f(x),$$

so that if the equation resulting from one or more applications of (*S*) or (*T*) to (1) has a zero invariant it may be solved in finite form. The solution of (1) is then obtained by a recurrence relation. The necessary and sufficient conditions that the above zero invariant exists are obtained.

More general transformations analogous to the one used by Lévy (*Journal de l'Ecole Polytechnique*, tome 38 (1886), page 67) are investigated and are found not to be generally useful in obtaining solutions.

6. The general class of integrals

$$J(x) = \int_0^\infty \varphi(t) \frac{g(x+t)}{g(x)} dt,$$

in which

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right),$$

includes several instances each of which is central in integral transformations of fundamental importance. Professor Carmichael points out that the fundamental convergence properties of these integrals are of simple character when $\varphi(t)$ and $g(x)$ are subject to certain broad general restrictions. Under these restrictions he develops the basic general elements of this convergence theory.

7. In this note Professor Lefschetz gives an account of a study of abelian manifolds from the point of view of their connectivity and their multiple integrals.

The results arrived at are closely related to some general theorems recently published by him along that line and also to the classical investigations on hyperelliptic surfaces of Picard, Humbert, and the geometers of the Italian school.

8. In a paper presented to the Society at the February meeting Dr. Rider developed a theory of the calculus of variations for n dimensions, using an integral of the form

$$\int_{t_0}^{t_1} f(x_1, \cdots, x_n, \tau_1, \cdots, \tau_{n-1}) \sqrt{x_1'^2 + \cdots + x_n'^2}\, dt.$$

The present paper gives the f_1-function for such an integral. It will be published in *Washington University Studies*.

9. Professor Wilczynski's paper involves a discussion of the various rational osculants determined by a given analytic function, the geometric relations between their zeros and poles, and a number of new notions closely related to intrinsic geometry.

10. In this note Mr. Rouse obtains a number of equivalent sets of conditions for linear dependence relative to linear integral forms of the type

$$\varphi_1(x) - \int_a^b k_1(x, y)\varphi_1(y)dy - \int_a^b k_2(x, y)\varphi_2(y)dy \equiv L(x)$$

which are analogous to the conditions for linear dependence which one finds in connection with the theory of linear algebraic equations. These conditions can easily be extended to the case of a system of m linear integral expressions in n functions.

11. It is known that the determinant D of order n whose n^2 elements are independent variables is invariant under a group G of order $(n!)^2$, and hence this polynomial assumes $n^2!/(n!)^2$ values when it is transformed by the substitutions of the symmetric group of degree n^2. These values may be arranged in pairs such that each pair is composed of polynomials which differ only with respect to sign and hence the square of D is transformed into itself by a substitution group K of order $2(n!)^2$. The main object of Professor Miller's paper is to study the groups G and K.

The groups G and K are simply isomorphic with imprimitive groups of degree $2n$ corresponding to the permutation of the rows and columns of D. When $n > 3$, G is a simply transitive primitive group and the subgroup composed of all its substitutions which omit a given element has two transitive constituents. The group $\underline{G}$ corresponding to the permutations of the positive terms of D under G is also a simply transitive primitive group but its subgroup composed of all the substitutions which omit a given element has a number of transitive constituents which increases indefinitely with n. The factors of composition of G are 2, 2, $n!/2$, $n!/2$, and hence the factors of composition of K are 2, 2, 2, $n!/2$, $n!/2$ and both G and K are unsolvable whenever $n > 3$.

12. The object of Professor Miller's second paper is to furnish a group theory proof for the following two well known theorems: Each of the sums obtained by adding $p-1$, $p-2, \cdots, 2$ figurate numbers of orders $2, 3, \cdots, p-1$ respectively is divisible by p whenever p is any odd prime number. Each of the coefficients except the first and the last in the expansion of $(a+b)^p$ is divisible by p. It is not claimed that the group theory proofs are easier than those usually given but they are based upon essentially different concepts and exhibit another element of contact between group theory and elementary number theory. The new proofs are based upon properties of the Sylow groups of order p^{p+1} contained in the symmetric group of degree p^2.

13. In this paper Professor Brand solves the systems composed of the differential equation of the elastic curve of a beam,

$$EIy'' = -M(x),$$

and the homogeneous boundary conditions $y(0) = 0, y'(0) = 0$ and $y(0) = 0$, $y(l) = 0$ by the use of Green's functions; he obtains two modified forms of the Fraenkel formula for flexural deflections which are particularly adapted to the study of statically indeterminate beams. After forming the solutions of the above equation with the non-homogeneous conditions $y(0) = \alpha$, $y'(0) = \alpha'$ and $y(0) = \alpha$, $y(l) = \beta$, the results are applied to determine the moments and reactions at the supports, and the deflections, of restrained and continuous beams under any system of loads. In particular, a very general form of Clapeyrou's theorem of three moments is obtained. The usual form of the Fraenkel formula, which has hitherto been deduced from the principles of mechanics, is shown to be a consequence of the above differential equation and its boundary conditions.

14. In this paper Professor Emch proves the theorem that every affinity in space, a collineation leaving the plane at infinity invariant, and defined in cartesian coordinates by

$$x' = a_0 + a_1x + a_2y + a_3z,$$

$$y' = b_0 + b_1x + b_2y + b_3z,$$

$$z' = c_0 + c_1x + c_2y + c_3z,$$

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independence. The author discriminates between an "invariant" and a "variable" type of independence of propositional functions $P_1, P_2, \cdots, P_n$ $(n = 2, 3, \cdots)$ with variable components $R_1, R_2, \cdots, R_i$ $(i = 1, 2, \cdots)$ of respective ranges $\rho_1, \rho_2, \cdots, \rho_i$. A second discrimination is that between "formal" and "material" independence. Subordinate to these discriminations is the characterization of the grade of independence by an index (n, k) $n \geqq k \geqq 0$ where k is the number of false propositions which arise when specific selections are suitably made from the variables $R_1, R_2, \cdots, R_i$. In this classification Professor E. H. Moore's "complete independence" appears as a variable independence of index (n, n) whereas the independence of Peano appears as an invariant independence of index $(n, 1)$.

In the second part of the paper the author calls attention to the use of "supernumerary" (non-independent) indefinables in primitive propositions as an instrument in the solution of problems of complete independence, i. e., if it is required to construct a completely independent set of postulates for a given logical domain the author would propose as a method for attaining this end a set of postulates involving supernumerary indefinables. Application to the author's postulates for an abstract group is made.

17. The paper of Professor Hildebrandt considers a number of generalizations of the theorem of Toeplitz which gives a necessary and sufficient condition under which if $\text{limit}_n\, x_n = x$ then

$$\text{limit}_n\, y_n = \text{limit}_n \Sigma_1^n a_{nm} x_m$$

is also x. These generalizations are a result of replacing the finite sum by an infinite sum, and considering under what conditions the sequence $\{y_n\}$ shall approach a limit, when the members of the sequence $\{x_n\}$ constitute a function belonging to various classes in the space of a denumerably infinite number of dimensions. The case in which x is a function of a continuous variable is also considered, the transformations from x to y being then on the basis of integration.

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A THEOREM ON THE VARIATION OF A FUNCTION.

BY DR. PAUL R. RIDER.

THE following is a well known theorem of differential geometry:

The differential quotient $d\phi/ds$ (ds is the element of arc) of a function $\phi(u, v)$ at a point on a surface varies in value with the direction from the point. It equals zero in the direction tangent to the curve $\phi = c$, and attains its greatest absolute value in the direction normal to this curve.*

This theorem admits of a generalization if we use a more comprehensive definition of length, a definition sometimes employed in the calculus of variations. Let then

$$S = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

be the generalized length of arc along a curve

$$(C) \qquad x = x(t), \quad y = y(t).$$

By reason of homogeneity conditions†

$$\begin{aligned} S &= \int_{t_0}^{t_1} F(x, y, \cos\theta, \sin\theta)\sqrt{x'^2 + y'^2}dt \\ &= \int_{s_0}^{s_1} F(x, y, \cos\theta, \sin\theta)ds, \end{aligned}$$

in which

$$\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin\theta = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

Then

$$\begin{aligned} A = \left|\frac{d\phi}{dS}\right| &= e\frac{\phi_x dx + \phi_y dy}{F(x, y, \cos\theta, \sin\theta)ds} \\ &= e\frac{\phi_x \cos\theta + \phi_y \sin\theta}{F(x, y, \cos\theta, \sin\theta)}, \end{aligned}$$

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* See Eisenhart, Differential Geometry, pp. 82–83.

† See Bolza, Vorlesungen über Variationsrechnung, p. 194.

where subscripts indicate partial differentiation, and where e is chosen equal to ± 1 so as to make A positive. Differentiating A with respect to θ, and setting the result equal to zero, we get

$$(1)\quad F(-\phi_x \sin\theta + \phi_y \cos\theta) \\ - (\phi_x \cos\theta + \phi_y \sin\theta)(-F_{x'} \sin\theta + F_{y'} \cos\theta) = 0,$$

$F_{x'}$, $F_{y'}$ denoting partial derivatives of F with respect to its third and fourth arguments respectively. Since

$$F = F_{x'} \cos\theta + F_{y'} \sin\theta,^*$$

equation (1) reduces to

$$(2)\quad \phi_y(x, y)F_{x'}(x, y, \cos\theta, \sin\theta) \\ - \phi_x(x, y)F_{y'}(x, y, \cos\theta, \sin\theta) = 0,$$

and if we define direction on the curve $\phi = c$ by means of the angle $\bar{\theta} = \arctan(-\phi_x/\phi_y)$, (2) becomes

$$F_{x'}(x, y, \cos\theta, \sin\theta)\cos\bar{\theta} - F_{y'}(x, y, \cos\theta, \sin\theta)\sin\bar{\theta} = 0.$$

But this equation determines the value of θ to which the curve $\phi = c$ is transversal.†

Therefore the differential quotient $d\phi/dS$ is equal to zero in the direction tangent to the curve $\phi = c$ and has its maximum absolute value in the direction to which the curve $\phi = c$ is transversal.

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TANGENTIAL INTERPOLATION OF ORDINATES AMONG AREAS.

BY DR. C. H. FORSYTH.

(Read before the American Mathematical Society December 27, 1917.)

IF we wish to interpolate several values in each interval between the successive ordinates $u_0, u_1, u_2, \cdots, u_n$ by finite differences, only a low order of differences can with propriety be used, since high orders based on ordinary statistical data lead to errors. But low orders of differences mean that few of the ordinates can be used at a time, and hence the curve passing through the values intermediate to u_1 and u_2 is not in general identical with the curve passing through the values intermediate to u_2 and u_3, and similarly for the rest of the intervals. It follows that in such successive interpolations the final and complete series of interpolated values have discontinuities at their points of intersection u_1, u_2, etc., which are sometimes quite serious.

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* See Bolza, loc. cit., p. 196.

† See Bolza, loc. cit., p. 303.

Osculatory interpolation represents an attempt to eliminate these discontinuities and succeeds for all practical purposes. It requires that whatever interpolation formula may be used as a basis, two intersecting interpolation curves passing through interpolated values in two adjacent intervals shall have the same slope and curvature at their point of intersection. Thus, the two curves are said to have a common osculating circle at this point and the modification of the basic interpolation formula is called the corresponding osculatory interpolation formula.

In recent years the most prominent statisticians have come to agree that differences higher than third or fourth are inappropriate for interpolating among ordinary statistical data and, as osculatory interpolation requires the use of fifth differences, formulas have been derived ignoring the idea of curvature and reducing the order required to third differences. However, in order to give full credit to the original plan even these third difference formulas are often referred to as "osculatory"; it has seemed more appropriate to the writer to refer to such formulas as "tangential" and the term will be used with that meaning in what follows.

In the December, 1916, issue of the *Quarterly Publications of the American Statistical Association* the writer gave the derivation of an interpolation formula which is to be used—explained geometrically—for interpolating ordinates among areas. For example, it may be used to estimate the population for individual ages from populations given in age groups. The formula to second differences is as follows:

$$(1)\quad u_{x/t} = \frac{w_0}{t} + (2x - t + 1)\frac{\Delta w_0}{t^2 \cdot 2!} + \{3x^2 + 3x(1 - 2t) + (1 - 3t + 2t^2)\}\frac{\Delta^2 w_0}{t^3 \cdot 3!}.$$

Thus, the individual value $u_{x/t}$ may be found from groups of t individual values—w_0, w_1, w_2, etc.—and their differences, where

$$w_0 = u_{0/t} + u_{1/t} + \cdots + u_{(t-1)/t}$$

and in general

$$w_x = u_{x+0/t} + u_{x+1/t} + \cdots + u_{x+(t-1)/t}.$$

The purpose of this paper is to modify formula (1) to give a "tangential" interpolation formula.

We assume the desired equation to be of the form

$$u_{(t+x)/t} = u_1 + ax + bx^2 + cx^3 + dx^4, \tag{2}$$

which evidently passes through the value u_1 for $x = 0$.

For $x = t$

$$u_2 = u_1 + at + bt^2 + ct^3 + dt^4.$$

Expressing u_1 and u_2 by formula (1), we have

$$u_2 - u_1 = at + bt^2 + ct^3 + dt^4 = \frac{\Delta w_0}{t} + (1 + t)\frac{\Delta^2 w_0}{2t^2}. \tag{A}$$

The derivative of (2) with respect to x becomes

$$D_x u_{(t+x)/t} = a + 2bx + 3cx^2 + 4dx^3, \tag{3}$$

and for $x = 0$

$$D_x u_1 = a. \tag{4}$$

But from formula (1)

$$D_x u_{x/t} = \frac{\Delta w_0}{t^2} + (2x - 2t + 1)\frac{\Delta^2 w_0}{2t^3}, \tag{5}$$

and for $x = t$

$$D_x u_1 = \frac{\Delta w_0}{t^2} + \frac{\Delta^2 w_0}{2t^3}. \tag{6}$$

Hence, by (4) and (6)

$$a = \frac{\Delta w_0}{t^2} + \frac{\Delta^2 w_0}{2t^3}. \tag{B}$$

Formula (5) may be expressed also in terms of the differences of u_1; hence, as in (6)

$$(7)\qquad D_x u_2 = \frac{\Delta w_1}{t^2} + \frac{\Delta^2 w_1}{2t^3}$$

$$= \frac{\Delta w_0}{t^2} + (2t+1)\frac{\Delta^2 w_0}{2t^3} + \frac{\Delta^3 w_0}{2t^3},$$

since

$$\Delta w_1 = \Delta w_0 + \Delta^2 w_0$$

and

$$\Delta^2 w_1 = \Delta^2 w_0 + \Delta^3 w_0.$$

Formula (3) for $x = t$ becomes

$$(8)\qquad D_x u_2 = a + 2bt + 3ct^2 + 4dt^3.$$

Hence, from (7) and (8)

$$(C)\quad a + 2bt + 3ct^2 + 4dt^3 = \frac{\Delta w_0}{t^2} + (2t+1)\frac{\Delta^2 w_0}{2t^3} + \frac{\Delta^3 w_0}{2t^3}.$$

We now have three equations (A), (B), and (C) out of four necessary equations to determine the values of the four coefficients a, b, c, and d in equation (2). Ordinarily, an equation of the third degree would be sufficient, in which case only three coefficients would be needed, but formula (1) is peculiar in the respect that when a group, say w_1, is broken up into the individual values $u_1, u_{1+1/t}, \cdots u_{1+(t-1)/t}$ their sum must finally equal w_1 and in establishing formula (2) this requirement must be maintained. This requirement furnishes the desired fourth equation. If we give x successively the values $0, 1, 2, \cdots, t-1$ in formula (2) and sum the results we obtain w_1 or $w_0 + \Delta w_0$. This yields

$$w_0 + \Delta w_0 = tu_1 + \frac{t(t-1)}{2}a + \frac{t(t-1)(2t-1)}{6}b$$

$$+ \frac{t^2(t-1)^2}{4}c + \frac{t(t-1)(2t-1)(3t^2-3t-1)}{30}d.$$

Transposing the term tu_1 to the left side and noting that by formula (1)

$$u_1 = \frac{w_0}{t} + (t+1)\frac{\Delta w_0}{2t^2} + (1-t^2)\frac{\Delta^2 w_0}{6t^3}$$

we have, on simplifying

$$(D)\quad a + \frac{(2t-1)}{3}b + \frac{t(t-1)}{2}c + \frac{(2t-1)(3t^2-3t-1)}{15}d = \frac{\Delta w_0}{t^2} + (t+1)\frac{\Delta^2 w_0}{3t^3}.$$

Solving the four equations (*A*), (*B*), (*C*), and (*D*) simultaneously, we obtain

$$a = \frac{\Delta w_0}{t^2} + \frac{\Delta^2 w_0}{2t^3},$$

$$b = \frac{\Delta^2 w_0}{2t^3} + \frac{3t^2-2}{4t^4(t^2+1)}\Delta^3 w_0,$$

$$c = \frac{1-4t^2}{2t^5(t^2+1)}\Delta^3 w_0,$$

$$d = \frac{5\Delta^3 w_0}{4t^4(t^2+1)}.$$

Finally, substituting these values in equation (2) and collecting terms we obtain

$$(9)\quad u_{(t+x)/t} = w + (2x+t+1)\frac{\Delta w}{2} + (3x^2+3x+1-t^2)\frac{\Delta^2 w}{6} + x^2(x-t)(5tx+2-3t^2)\frac{\Delta^3 w}{4t(t^2+1)},$$

where, for brevity, we let $\Delta^n w = \Delta^n w_0/t^{n+1}$ (and $w = w_0/t$).

In applying (9) to, say the group w_1, it is to be noticed that the slope at u_1 ($= u_{t/t}$)—by (6)—is determined by w_0, w_1, and w_2 (to be referred to for the present as "the first set") and at u_2 ($= u_{2t/t}$)—by (8)—by w_1, w_2, and w_3 (to be referred to as "the second set"). Then, in applying the formula to the next group w_2, its first set (used to determine the slope at u_2) is identical with the second set of the preceding group and the interpolation curves of the two groups have the same slope at their point of intersection u_2; and similarly for u_3, u_4, etc. The idea of curvature could be added if desired in exactly the same way if higher differences were proper.

When, as often happens, $t = 5$, (9) becomes

$$(10)\quad u_{(5+x)/5} = w + (x + 3)\Delta w + (x^2 + x - 8)\frac{\Delta^2 w}{2} + x^2(x - 5)(25x - 73)\frac{\Delta^3 w}{(104)5^2}.$$

It is seldom desirable to use either (1) or (9) to determine a single individual value—they are generally used to break up a group of values completely. In the latter case, those familiar with the use of finite differences know that much labor can be saved by using the corresponding formulas for the leading term and differences of the individual values. The application of such formulas is much shorter on the average than the repeated application of either (1) or (9) and the majority of the computation consists in mere addition.

If we let x equal successively $0, 1, \cdots, t - 1$ in formula (9) and difference the results four times we obtain the following leading term and differences

$$u_1 = w + \frac{t + 1}{2}\Delta w + \frac{1 - t^2}{6}\Delta^2 w,$$

$$\delta u_1 = \Delta w + \Delta^2 w + (1 - t)(2 + 5t - 3t^2)\frac{\Delta^3 w}{4t(t^2 + 1)},$$

$$\delta^2 u_1 = \Delta^2 w + \{(2 - 3t^2)(6 - 2t) + 5t(14 - 6t)\}\frac{\Delta^3 w}{4t(t^2 + 1)},$$

$$\delta^3 u_1 = \{6(2 - 3t^2) + 30t(6 - t)\}\frac{\Delta^3 w}{4t(t^2 + 1)},$$

$$\delta^4 u_1 = t\ 5!\Delta^3 w,$$

where, it is to be remembered, $\Delta^n w = \Delta^n w_0/t^{n+1}$ (and $w = w_0/t$).

When t has the value 5 the third differences prove relatively difficult to compute owing to the presence of a "13" in the denominator. If in the difference δu_1, $2.7\Delta^3 w/13$ is replaced by $2.6\Delta^3 w/13$ and in all the rest $\Delta^3 w/13$ is replaced by $\Delta^3 w/12$ the leading term and differences (for $t = 5$) become

$$\begin{aligned}
u_1 &= w + 3\Delta w - 4\Delta^2 w,\\
\delta u_1 &= \Delta w + \Delta^2 w + .2\Delta^3 w,\\
\delta^2 u_1 &= \Delta^2 w - .2\Delta^3 w,\\
\delta^3 u_1 &= - .6\Delta^3 w,\\
\delta^4 u_1 &= + .002\Delta^3 w_0,
\end{aligned}$$

where, it is to be remembered, $\Delta^n w = \Delta^n w_0/5^{n+1}$ (and $w = w_0/5$). It should be noticed, however, that $\delta^4 u_1$ is expressed more simply by $.002\Delta^3 w_0$.

The arbitrary changes just suggested give an error of only about one half of one per cent of $\Delta^3 w_0$ in w_1 (the sum of the individual values interpolated). The error in each individual value would then be much less and if only two more decimal places are used than are to be finally retained—which are all that are ordinarily necessary as found by experience—the errors would not appear at all in the results. As a check upon the work the sum of the individual values interpolated should be w_1 as given originally.

It should be pointed out that the formulas derived above can not be used for "end" values; that is, if the groups of values were $w_0, w_1, w_2, \cdots, w_n$ the formulas could not be used for interpolating or breaking up w_0 or w_n, for the derivation of the formulas is based upon the use of four values (w_0, w_1, w_2, and w_3) to break up w_1; that is, there must always be at least one group preceding the group to be broken up. To break up "end" values formula (1) could be used or the formulas for the leading term and differences to be found in the article cited.

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NON-EUCLIDEAN GEOMETRY.

Geometrical Researches on the Theory of Parallels. By N. LOBACHEVSKI. Translated from the original by G. B. HALSTED. New edition. Chicago and London, Open Court, 1914. 8vo. 50 pages. Cloth, price $1.25.

NON-EUCLIDEAN geometry had two independent discoverers: Johann Bolyai (1802–1860), a Hungarian officer in the Austrian army; and Nicolaus Lobachevsky (1793–1856), son of a Russian peasant, and graduate, professor, and rector, of the University of Kasan.*

As early as 1823 the former had grasped the real nature of his problem, and in 1829 he sent a completed manuscript on the subject to his father, Wolfgang Bolyai, who was a professor of mathematics, physics and chemistry. In 1832 the father published a Latin translation of the manuscript as an appendix* to the first volume of one of his scientific works.

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* The best history of non-euclidean geometry is by Bonola. It was translated into English and edited by H. S. Carslaw (Chicago, 1912).

It seems to have been after 1823 that the development of non-euclidean geometry occurred to Lobachevsky. This may be inferred from a passage in his book on Elementary Geometry:† "Up to the present no one has been able to find a rigorous proof of the fifth postulate of Euclid; what has been given we may only call explanations; these do not deserve to be characterized as mathematical proofs in the true sense." The first fruit of new study in a geometry independent of Euclid's hypothesis was a paper read before the physico-mathematical section of the University of Kasan in 1826. This paper does not seem to have been preserved, but the essential parts of it were contained in a memoir published in 1829–30. Lobachevsky indicated further applications of the new theory in three more Russian memoirs of 1835, 1835–38, 1836 ("Géométrie Imaginaire" which appeared in *Crelle's Journal* for 1837 is really a free translation of the memoir of 1835). He endeavored to call the attention of mathematicians to his discoveries by publishing a summary of his results in a little book which appeared at Berlin in 1840 under the title: Geometrische Untersuchungen zur Theorie der Parallellinien.‡ It is this little work which Dr. Halsted has here given us in English form.

It is probable that Lobachevsky never even heard of Bolyai, who, in 1848, made himself thoroughly acquainted with the Geometrische Untersuchungen.

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* A facsimile edition was published at Maros-Vásárhelyini in 1907. 4 + 28 pp

† The manuscript of this work was sent to St. Petersburg in 1823 but failed of publication at that time on account of the unfavorable report of the academician N. Fuss. In 1898 the manuscript was discovered in the archives of the university of Kasan and it was published in 1909 by the Physico-Mathematical Society of Kasan (8vo. 3 + 67 pp.). Titles of sections indicate that Méray's ideas of fusion were anticipated, many years before, by Lobachevsky: Measuring lines; Angles; Perpendiculars; Measuring solid angles; Regular polygons and polyhedrons; Equivalent triangles; Measuring right angles, triangles and other figures; Parallelograms; Measuring of pyramids and polyhedrons; Measuring the circumference and the area of a circle, the volume of the cylinder and cone; Measuring the area of the right cylinder and right cone; On the magnitude of the volume and surface of the sphere.

‡ Facsimile reprint, Berlin, 1887; [Geometrical works of Lobachevsky] (in Russian), Kasan, vol. II, 1886, pp. 553–578. French translation by J. Hoüel in *Mém. de Bordeaux*, tome 4 (1866), also separately (2d ed. 1895), and in Recherches géométriques sur la théorie des parallèles, Paris, 1900.

There is little doubt that Carl Friedrich Gauss (1777–1855) was the first to have a clear view of a geometry independent of the fifth postulate; but this fact "was only revealed after the works of Lobachevsky (1829–30) and J. Bolyai (1832) appeared." To these geniuses alone, by reason of priority of publication, must belong all honor due to those confident of their results and unafraid of the world's criticism.

In February, 1832, Wolfgang Bolyai sent a copy of his son's work to Gauss. In acknowledging its receipt Gauss wrote, on March 6,* somewhat as follows:

". . . Now something concerning the work by your son. If I commence with this, '*that I can not praise it,*' you may well be taken aback for a moment, but I can not do otherwise; to praise it would be to praise myself: for the whole content of the work, the path which your son has followed, and the results to which he is led, agree almost throughout with my own meditations which have occupied my mind for 30–35 years. So I am surprised to the utmost. My intention with respect to my own work, concerning which little has been put on paper up to the present, was to allow nothing to be known of it during my life time. Most people do not have any appreciation of what is involved in the subject, and I have found only few men who took up with any particular interest what I communicated to them concerning it.' To be able to take such an interest it is first of all necessary to have devoted careful thought to the real nature of what is lacking, and upon this matter almost all men are far from clear. On the other hand, it was my intention some time to put the whole matter on paper, so that it should not one day perish with me.

"Greatly have I been surprised that I am spared this trouble and much do I rejoice that it is just the son of my old friend who has got the start of me in such a remarkable manner."

And then follow a discussion of details and a presentation of his own development of some of the parts.

In a letter to Gerling about three weeks earlier Gauss characterized the younger Bolyai as a "genius of the first order." He wrote as follows:

". . . Noch bemerke ich, dass ich dieser Tage eine kleine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle *meine eigenen Ideen und RESULTATE* wiederfinde, mit grosser Eleganz entwickelt, obwohl in einer für jemand, dem die Sache fremd ist, wegen der Concentrirung etwas schwer zu folgenden Form. Der Verfasser ist ein *sehr* junger österreichischer Officier, Sohn eines Jugendfreundes von mir. . . . Ich halte diesen jungen Geometer v. BOLYAI für ein Genie erster Grösse. . . ."

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* Gauss, Werke, Bd. 8, Göttingen, 1900, pp. 220–221.

Gauss was not only familiar with all the writings of Lobachevsky referred to above but he possessed all of them by 1844, and indeed, probably a year or so earlier, when he was responsible for having Lobachevsky elected as correspondent of the Royal Society of Sciences at Göttingen. Contrasting the Geometrische Untersuchungen with the memoirs, he wrote to Gerling on February 8, 1844:* "Namentlich hat sie viel mehr Concinnität und Präcision, als die grösseren Aufsätze des Lobatschewsky, die mehr einem verworrenen Walde gleichen, durch den es, ohne alle Bäume erst einzeln kennen gelernt zu haben, schwer ist, einen Durchgang und Uebersicht zu finden."

In a letter to his friend Schumacher in 1846 Gauss wrote further† concerning the Untersuchungen: ". . . materiell für mich Neues habe ich also im Lobatschewskyschen Werke nicht gefunden, aber die Entwickelung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von Lobatschewsky auf eine meisterhafte Art in ächt geometrischem Geiste. Ich glaube Sie auf das Buch aufmerksam machen zu müssen, welches Ihnen gewiss ganz exquisiten Genuss gewähren wird. . . ."

The ideas inaugurated by Lobachevsky and Bolyai did not obtain wide recognition for many years, indeed for more than a decade after Gauss's death. Shortly after Hoüel had published in 1866 the French translation of the Untersuchungen and "Appendix" together with the commendatory passages from Gauss's correspondence, the works of the almost unknown Slav and Hungarian mathematicians came into their own, and non-euclidean geometry was taken up as a serious study.

The work before us for review is described as a "new edition." The translation seems to have been originally published,‡ with "Translator's preface," at Austin, Texas, in *Scientiæ Baccalaureus* for February, 1891.§ A new edition with type reset, with the same preface now dated May 1, 1891, with translator's introduction and with translator's appendix was published as a Bulletin of the University of Texas.* This same edition with date 1891 and with a "Note to the fourth edition," was apparently published in 1892. But "The Nemonic Series, Vol. IV" is also called the fourth edition though listed by Sommerville as the third. Halsted's translation was also published in the *Tōkyō Sūgaku Butsurigaku Kwai Kiji* (Proceedings of the Physico-mathematical society of Tokyo).†

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* Gauss, Werke, Bd. 8, p. 237.

† Gauss, Werke, Bd. 8, p. 238.

‡ D. M. Y. Sommerville (Bibliography of Non-Euclidean Geometry, 1911) lists an edition published at Rolla, Mo., in 1891.

§ Vol. 1, no. 3, pp. 123–164.

Except as to binding, the edition before us is practically identical with that issued by the University of Texas. Indeed the same plates seem to have been used except for pages 5, 9 and 10. Small changes have been made on pages 7, 19, 47, 48. Pages 48 and 49 of the translator's preface have been replaced by paragraphs headed "Bibliography" and the portrait frontispiece of Lobachevsky, which has been added, seems to have been reproduced from Engel's volume.‡

It seems a pity that a little more care had not been exercised in this final revision. On page 6 we still find Henricus Billingsly instead of Henry Billingsley. While the following sentence may have been true in 1891, the date of the preface, it does not describe the conditions in 1914, the date of the volume. "Even to-day in the vast system of examination set by Cambridge, Oxford and the British government, no proof will be accepted which infringes Euclid's order, a sequence founded upon his set of axioms." Figures 13, 15, 24, 28, 31, 32, 33, still differ in some respects from Lobachevsky's originals. For example these had: lines *EG* and *EH* in Fig. 13; no arc *C'B'* in Fig. 15; *LK* as bisector of *HC* in Fig. 24 and so on.

The translation is faithfully and skilfully done. While the paragraphing is quite different from the original the change has possibly contributed to clearer presentation. The long and unnecessary spacings after sections 16 and 24 do not occur in the original.

The Open Court publishing company has rendered a service to mathematicians in placing at their disposal English editions of Lobachevsky's and Bolyai's masterpieces.

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* Austin, 1891, 50 pp.

† Vol. 5 (1892–94), pp. 6–50.

‡ Nikolaj Iwanowitsch Lobatschefskij. Zwei geometrische Abhandlungen aus dem Russischen uebersetzt, mit Anmerkungen und mit einer Biographie des Verfassers. Von F. Engel. Leipzig, 1898. Reviewed by F. S. Woods in this BULLETIN, vol. 6, 1900, pp. 339–344; 443–452.

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FRENCH SCIENCE.

Science and Learning in France with a Survey of Opportunities for American Students in French Universities. An Appreciation by American Scholars. The Society for American Fellowships in French Universities, 1917. 8vo. 40 + 454 pages. (For sale by A. C. McClurg and Co., Chicago: cloth, $1.50; stiff paper, $1.00.)

Exposition Universelle et Internationale de San Francisco. La Science Française. 2 tomes. Paris, Ministère de l'Instruction Publique et des Beaux Arts, 1915. 397 + 405 pages.

L'Histoire des Sciences et les Prétentions de la Science Allemande. By E. PICARD. Paris, Perrin et Cie., 1916. 12mo. 49 pages. Boards. Price 60 centimes.

THE Society for American Fellowships in French Universities has announced that it is prepared to award ten or more fellowships, each of one thousand dollars annually, to American graduate students who undertake to study in French Universities. The fellowships will be given for studies in any of the following fields of knowledge: anthropology, archæology, astronomy, botany (and agriculture), chemistry, criminology, education, engineering, geography, geology, history, law, mathematics, medicine, classical philology, oriental philology, romance philology, semitic philology, English philology, philosophy, physics, political science (including economics and international law), psychology, religion, sociology, and zoology (including biology).

The awards will be based on nation-wide competition. It is planned to make the first annual award in the spring of 1918, so that the Fellow may depart for France in July, 1918, if the war has ended by that time. It has been announced that a circular of information, giving details as to the persons eligible, the form of application, and the mode of making the award, will be ready for distribution to all American institutions by October, 1917. Each fellowship is to be held for two years, and new fellowships will be awarded each year for ten years at least.

Preparatory to putting this scheme into effect the Society has published a sumptuous handbook for American graduate students in French universities. This is the work under review. It was edited by Professor John H. Wigmore, of Northwestern University.

In the Introduction Charles W. Eliot writes on "The Mind of France," and George E. Hale, a correspondent of the Institute of France, tells of "The Intellectual Inspiration of Paris." Then follow 33 chapters, each prepared by a drafting committee of specialists and each illustrated by a portrait of at least one Frenchman distinguished in the subject treated. "Each chapter," to quote from the preface, "sets forth briefly

"1. The progress of French science and learning in the particular field during the last half century or so; the notable achievements; the eminent leaders; the special lines of development; in general, the share in the world's accomplishments;

"2. The courses of instruction given, now or recently, at the universities of France, particularly at the University of Paris; the names of the most important scholars, with a mention of their principal works and of the special fields of research over which they preside;

"3. The facilities available for study and research, including libraries, laboratories, archives and museums, the auxiliary institutes, special schools, and learned societies and committees.

"There is also an Appendix describing the methods of instruction and study, the standards of preparation expected of the student, the system of degrees, the customs as to residence, attendance, etc.; the regulations as to fees, etc.; and other facts useful to the visiting student." The appendix is illustrated by pictures of buildings in the University of Paris, etc.

The drafting committee for the chapter on Astronomy consisted of: P. Fox, G. E. Hale, F. R. Moulton, W. D. MacMillan, and H. N. Russell; for the one on Engineering: I. N. Hollis, H. M. Howe, A. C. Humphreys and A. Sauveur; for the one on Mathematics: D. R. Curtiss, T. F. Holgate, E. H. Moore, and E. B. Wilson; and for the chapter on Physics: H. Crew, A. A. Michelson, and W. C. Sabine.

The committee on mathematics finds that this science "has always made a special appeal to French genius distinguished by its fondness for logic and its striving for perfection in form," and it pays tribute to the genius of Henri Poincaré, of whom "at his death in 1912 it was the universal verdict that he must be considered the greatest mathematician of his age." The portrait of Poincaré reproduced in the chapter was taken about thirty years ago. It is the same one that was reproduced in the *American Journal of Mathematics* for 1890.

A sentence on page 357 seems to indicate that the writer does not know that the Sorbonne is only that part of the University of Paris in which instruction is given in the faculties of sciences and letters.

The volume is of extraordinary interest.

On account of the war, France's exhibit at the San Francisco exposition of her scientific achievements was of an unusual nature. It consisted of a library in which were collected periodicals and books, ancient and modern, illustrating her scientific progress and activity. "Pour chaque science," President Poincaré writes in the introduction, "Pour chaque science, on a essayé de remonter au moment où, en France, un ordre d'études, importantes par le profit intellectuel ou moral qu'elles procurèrent aux hommes, fut abordé pour la première fois et devint l'objet de recherches systématiquement conduites. On a voulu marquer l'origine, le point d'où sont partis tant de hardis explorateurs pour l'éternel voyage à la recherche de la vérité; on a indiqué, sur les chemins tracés par leurs glorieux efforts, les sommets d'où ont été aperçus de nouveaux horizons; on a signalé enfin, avec quelque insistance, l'étape actuellement atteinte qui sera dépassée par le travail de demain poursuivi dans des direction que l'on a cherché à préciser.

"On verra donc dans la bibliothèque des livres vénérables et illustres par où une grande idée fut semée dans le monde; puis les publications principales, grâce auxquelles les rameaux puissants d'une doctrine se développèrent; enfin, pour l'heure présente, un choix assez large d'ouvrages individuels ou de recueils collectifs où l'on trouvera la preuve tangible de l'activité scientifique de la France, et où l'on pourra apprécier la luxuriante floraison produite par une habile culture."

In order that the visitor might better appreciate the choice of works, two volumes of admirable articles were written by distinguished specialists to summarize France's contributions to each science. These articles were each concluded by a bibliography, which is quite extensive in some cases.* Each is illustrated by an excellent portrait of some deceased master in the field considered.

The 14 articles in the first volume are on: philosophy (by Bergson, 23 pages), sociology, science of education, mathematics (by Appell, 15 pages), astronomy (by Baillaud, 37 pages), physics (by Bouty, 21 pages), chemistry, mineralogy, geology, paleobotany, zoological paleontology, biology, the medical sciences, the science of geography. The 19 articles of the second volume deal particularly with the sciences of archaeology, history, art, language, and philology.

In the article on mathematics there are six main headings: arithmetic, algebra, mathematical analysis, geometry, kinematics and mechanics. Under astronomy there are three parts, one on the seventeenth century, a second on the eighteenth and the first part of the nineteenth century, the third on contemporary astronomy.

The volumes should be in every college library.

The motto for Picard's little book is by Etienne Lamy, secrétaire perpétuel de l'Académie française. It is as follows:

"Les Allemands ont pris pour devise: L'Allemagne au-dessus de tout.

"Nous ne répondons pas: La France au-dessus de tout.

"Une seule devise est digne de la France: Au-dessus de tout, la vérité."

Picard confines his attention to the mathematical, physical and natural sciences. He shows that few indeed are the contributions of Germans toward their development, in comparison with those by scholars of other countries. While an attempt is made to explain the psychology of the German's "au-dessus de tout" illusion, the great discoveries of such men as Gauss are not slighted by any means. As far as it goes, the survey is very satisfying.

But the intellectual triumphs of France along these lines are much more fully set forth in the volumes whose plans have been described above. One of them, happily, it has been America's privilege to prepare. R. C. ARCHIBALD.

BROWN UNIVERSITY, *July*, 1917.

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* In the article on mineralogy, for example, about fifteen pages. Books whose titles are marked with an asterisk in these bibliographies were to be seen in the library exhibit.

FUNCTIONS OF TWO COMPLEX VARIABLES.

Lectures Introductory to the Theory of Functions of Two Complex Variables. By A. R. Forsyth. Cambridge University Press, 1914. xvi + 281 pp.

The present volume consists substantially of a course of lectures delivered in the University of Calcutta in January and February, 1913, in response to a special invitation of the authorities accompanied by a stipulation that the lectures should be published. What was desired was an exposition of some subject that might suggest openings to those who had the will and the skill to pursue research. In accordance with this wish the author selected the theory of functions of two complex variables, a subject still in a preliminary stage of its development and one into the exposition of which he could incorporate a considerable body of results of his own.

No attempt is made to give a systematic discussion of the whole subject nor is attention concentrated upon one particular issue. Several distinct lines of investigation are dealt with, even though this required that their treatment should be relatively brief. The essential purpose throughout was to deal with a selection of principles and of generalities belonging to the initial stages of the theory of functions of two complex variables; and this was accompanied by the desire to establish some new results and to suggest some new problems of investigation.

The substantial results of the theory of functions of a single complex variable are assumed to be so familiar to the reader that only brief and indirect reference to them will usually suffice. Almost everywhere in the exposition the number of independent variables is restricted to two. Many of the propositions may readily be modified so as to apply to the case of n variables; but this is not always true. Consequently we have on the one hand a range of results which belong essentially to functions of more than one variable and on the other hand another range of results belonging essentially to functions of just two variables. Typical of the latter is the theory of quadruply periodic functions of two variables.

At several places the author has departed from the usual custom of dealing with only a single function of two complex variables and has considered simultaneously two such functions. Certain characteristic properties are in this way brought into fuller discussion and interesting results emerge which otherwise would not be obtained, particularly in the theory of quadruply periodic functions of two variables. He is led to the simultaneous investigation of two functions by the following considerations (pages 2–4): In the theory of functions w of a single variable z we have a relation of the form $f(w, z) = 0$ so that we may consider w as a function of z or z as a function of w, thus having a complete dual notion of inversion. But if w is a function of two complex variables z and z' we do not have such inversion in the complete sense. Consequently a second function w' is introduced. Then we have relations

$$F(w, w', z, z') = 0, \qquad G(w, w', z, z') = 0,$$

so that we may look upon w and w' as functions of z and z' or conversely upon z and z' as functions of w and w'.

This way of introducing the duality certainly leaves something still to be desired. There is nothing in the forms of the foregoing equations to suggest the pairing of w and w' on the one hand and of z and z' on the other. Moreover, if we are interested primarily in a single function of two variables we certainly have an added difficulty when we introduce another essentially unrelated function of the same two variables. It can hardly be expected that two such functions will throw essential light each upon the properties of the other. Only certain more general characteristics of a function can be expected to emerge in this way. This undoubtedly is valuable, but it also leaves something further to be desired in the way of information about the individual function primarily in consideration.

A modification of the method employed by the author seems to obviate both of these difficulties. Instead of considering only two essentially unrelated functions w and w' of the variables z and z', let us also treat in further detail the following special case, namely:

$$w = f(z, z'), \quad w' = f(z', z).$$

Under easily derived appropriate conditions which we shall not take space to state, it is clear that z and z' are inversely functions of w and w' of the form

$$z = f_1(w, w'), \quad z' = f_1(w', w).$$

We thus retain the complete duality of inversion which is so important in the case of functions of a single variable; and this is accomplished without the introduction of an essentially new function. Moreover, the pairs w, w' and z, z' play a rôle different from that of other pairs of pairs of the four quantities w, w', z, z'. All the results obtained by considering the two functions w and w' state properties belonging essentially to w alone, since w and w' are so simply related.

The first chapter deals principally with three methods of geometrical representation of two complex variables; namely: representation by points in four-dimensional space, representation by lines in ordinary three-dimensional space, representation by points in two planes. The conclusion is reached (page 14) that only two of these are even fairly useful, namely, the first and the last. To supply fully our needs in respect of geometric representation we must have some uniquely effective new idea. While awaiting its appearance we must be content with such imperfect representations as the above-named methods afford.

In the first chapter one finds also a definition of functions of two complex variables similar to the Riemann definition for the case of one variable.

In the second chapter we have a treatment of the linear transformation of complex variables:

$$w = \frac{az + bz' + c}{a''z + b''z' + c''}, \quad w' = \frac{a'z + b'z' + c'}{a''z + b''z' + c''}.$$

Such a transformation is reducible to one or another of three canonical forms according as the characteristic equation in θ, namely,

$$\begin{vmatrix} a - \theta & b & c \\ a' & b' - \theta & c' \\ a'' & b'' & c'' - \theta \end{vmatrix} = 0,$$

has three simple roots, one double root and one simple root, or one triple root. Invariant centers of the transformation are discussed. A special investigation is also given of the sort of frontiers in four-dimensional space which are left invariant as to quality by these transformations, the result corresponding to invariancy of circles under linear transformations in a single variable. If one puts $z = x + iy$, $z' = x' + iy'$, x, y, x', y' being real variables, then the invariant frontiers in question are defined by equations of the form

$$A(x^2 + y^2) + B(x'^2 + y'^2) + C(xx' + yy') + D(xy' - yx') + Ex + Fy + Gx' + Hy' = K,$$

where the symbols $A, \cdots, K$ denote constants. There is also a treatment of the invariants and covariants of quadratic frontiers, and of periodic transformations.

It seems to the reviewer that a considerable number of readers will be repelled from the book by its first two chapters unless they observe the relation, or rather for the most part the lack of relation, of these two chapters to the remainder of the work. They are in large measure a reproduction of two papers by the author. Their general interest is certainly less than much of that which follows, both of the author's researches and of the matter taken from the literature. Fortunately but little of these chapters is used in building the succeeding portions of the book. Consequently it would seem well to have a list of portions suited to a first reading, similar to such a list in Forsyth's Theory of Functions of a single variable. We venture to suggest such a selection, our primary aim being to indicate to the reader how he may readily get into the meat of the subject.

In the first chapter one may take sections 1–6, 14–16. The second chapter may be omitted entirely on a first reading, or one may take sections 21–27 so as to have in mind the first properties of linear transformations. Thus after reading eleven pages from the first chapter (and possibly eleven pages from the second chapter), one is ready to enter upon the main features of the subject at the beginning of the third chapter on page 57. From this point forward one may read Chapters III, IV, V, Chapter VI up to page 169, Chapter VIII (with reference to one or two parts of Chapter VII). The remainder of the book may then be taken in any order desired. It appears to the reviewer that the reader is likely to find the book more interesting if he follows a plan of selection similar to that just indicated.

In the third chapter begins the essential development of the theory of single-valued analytic functions of two complex variables. The problem here treated is that of their expressibility in power series. Certain of the fundamental theorems for functions of a single variable are readily carried over to the case of functions of two variables; as, for instance, those connected with the Taylor and the Laurent series expansions, dominant functions, approach to every value, and certain matters concerning analytic continuation. But in the classification of singularities an essential difference emerges. There are three types of singularities; to these are given the names *pole, unessential singularity, essential singularity*. Let k, k' be a singular place for the single-valued function $f(z, z')$. If no power series $P_0(z - k, z' - k')$ in non-negative powers of $z - k$, $z' - k'$ exists such that the product

$$P_0(z - k, z' - k')\cdot f(z, z')$$

is expressible in a series of non-negative powers of $z - k$, $z' - k'$, then k, k' is called an essential singularity of $f(z, z')$. If the foregoing product can be formed so as to afford a function $P_1(z - k, z' - k')$ expansible in non-negative powers of $z - k$, $z' - k'$, then $f(z, z')$ is said to have a pole or an unessential singularity at k, k' according as it is or is not possible to choose $P_0(z - k, z' - k')$ so that $P_1(0, 0)$ shall be different from zero.

The fourth chapter is devoted principally to a consideration of the classic theorems of Weierstrass concerning the representation of a single-valued function in the vicinity of any of its various places, whether ordinary or singular. But little is said of the case of an essential singularity. For all other places the matter rests on the representation of a function in the vicinity of a zero of the function, as one sees readily from the definitions stated in the preceding paragraph. The concept of the divisibility of one function by another and the consequent theory of reducibility are also treated in this chapter. On pages 83 and 84 the author implicitly makes essential use of certain results in the theory of divisibility, whereas the theory is itself first developed on pages 112 ff.

In Chapter V one meets the following (and related) theorems analogous to corresponding ones for functions of a single variable: A function without essential singularities (either in the finite region or at infinity) is a rational function; a function which has no essential singularity in the finite region is expressible as a quotient of two functions each of which is without any singularity in the finite region.

The sixth chapter is devoted to integrals. In the first part is given a treatment of double integrals of uniform functions of two variables. Here one finds Poincaré's extension of Cauchy's main integral theorem, followed by several simple examples of a subject which awaits further development. The latter part of the chapter is concerned with integrals, both single and double, of algebraic functions, a theory to which Picard has made fundamental contributions. The author here takes the line of introducing two algebraic functions of two variables, thus following up (for the first time except for his treatment of linear transformations) his notion of the simultaneous consideration of two functions. But little more is done than to give an introduction to this problem, particularly in its preliminary formal aspects. Reference is made to Picard and Simart's Théorie des Fonctions algébriques de deux Variables indépendantes, where the case of a single function of two variables is treated in illuminating detail. The opinion is expressed that the further simultaneous consideration of two functions will lead to interesting extensions of the theory.

The seventh chapter deals with the so-called "level places" of two simultaneous single-valued functions of two variables, each function being without an essential singularity in the finite region. It is shown that two such functions simultaneously approach zero for a suitable approach of z and z' separately to suitably chosen points in their respective planes of variation, these planes being understood to contain the point infinity. Such a point is called a common zero of the two functions. If the two functions are further restricted to be independent and without common factors, then their common zeros are isolated. These results are readily applied to a consideration of the "level places" of two functions $f(z, z')$ and $g(z, z')$, namely, the common zeros of $f(z, z') - \alpha$ and $g(z, z') - \beta$, where α and β are constants.

The eighth and last chapter is devoted to the theory of single-valued periodic functions of essentially two variables. Such a function cannot have more than four linearly independent period-pairs. The cases of one period-pair and of two period-pairs are degenerate. Type forms of period-pairs for the non-degenerate cases of three and of four period-pairs are worked out in detail. The method is somewhat tedious and probably could not be extended to the case of functions of n variables; but it brings into relief the character of the various cases which initially come into consideration. The theta-functions of two variables are employed in the development of the theory. Theorems concerning algebraic relations among homo-periodic functions are derived. Simple examples of hyperelliptic functions are introduced. But the author has not attempted to expound the details of the theory of periodic functions of two variables; he has left that to the specific treatises on this subject.

A few minor misprints may be mentioned: "real" for "pure imaginary" in line 15 of page 32; "not universally" for "indeed never" in line 16 up on page 83; obvious misprints in footnote on page 75, in line 16 up on page 85, in line 15 on page 126, in line 5 up on page 236.

On the whole the book is well written and the exposition is usually clear. Some (but not many) infelicities of expression occur of which the following are examples: The author speaks several times (as on page 50) of "invariance under a single transformation only"; he employs repeatedly the redundant description "uniform, continuous and analytic" (see an instance on page 91 and the definition of "analytic" on page 59); he speaks sometimes (as on page 60) of a fixed domain as if it might be infinitesimal in extent. On page 241 and elsewhere in the following discussion it is desirable to specify *uniform* convergence as well as *absolute* convergence.

It is desirable to have an explicit definition of the region at infinity; this is not supplied by the author. It is assumed without remark (page 58 and elsewhere) that the infinite region in the z-plane and also that in the z'-plane is the infinite region of the usual complex plane; and then the infinite region for z and z' simultaneously is that in which either z or z' is infinite.

Probably minor defects of this sort (of which we do not attempt a complete list) will interfere with nothing but the comfort of the reader likely to be interested in this subject and will not seriously impair for him the value of the book.

We add a remark concerning a choice deliberately made by the author, a choice particularly unfortunate in the opinion of the reviewer.

Consider the function $e^{1/x}$. For $x = 1$ this function has the value e. When x approaches the value 1 in any way the function approaches the value e. We may say that the function possesses or acquires the value e at the point $x = 1$. Now consider what happens to the function as x approaches zero. It is easy to show that the function approaches e if x approaches zero *in a proper manner*. It is also equally easy to show that the function approaches any preassigned value if x approaches zero *in a suitable manner*. We may then say that $e^{1/x}$ *approaches* e (or any other assigned value) in case x approaches zero in a way appropriately dependent on this assigned value. Forsyth would still speak of the value e as a value which the function *acquires* or *possesses* at the point $x = 0$, using precisely the same terms as for the case of x approaching 1 and deliberately rejecting the distinction which clearly exists. See especially pages 77 and 78.

To the reviewer it seems definitely desirable to make the distinction and to mark it by a suitable choice of terminology, using *acquires* or *possesses* in the one case and *approaches* in the other. Some such distinction is generally made. There is a definite loss and no gain in refusing to make it, as most readers will probably be led to suspect by an examination of Forsyth's remarks (pages 77–79) about Picard's theorem to the effect that a non-constant entire function $f(z)$ acquires every value with at most a single exception.

Many of the results stated in the book need to be interpreted in the light of the fact that the author refuses to make the distinction in question. We illustrate by a single example. On page 198 we have the theorem: "Two independent functions, regular throughout the finite part of the field of variation, vanish simultaneously at some place or places within the whole field." Here "vanish" means to "acquire" the value zero in the sense of Forsyth's use of the term. What is actually proved is that the two functions simultaneously approach zero for an appropriate approach of z, z' to some appropriately chosen place k, k', finite or at infinity. The theorem does not look so attractive when clothed in this dress; but its true character is more readily recognized.

Similar remarks may be made about not a few other theorems. It is unfortunate for the reader to have to look through the proof of a theorem to see which sense of "acquire" gives to the theorem the best or largest meaning which the proof allows.

Finally, let us say that there can be no doubt that Forsyth has rendered his colleagues a distinct service in adding this book to his already long list of useful publications. It will be of definite value to a large number of persons interested in the theory of functions of complex variables.

R. D. CARMICHAEL.

SHORTER NOTICES.

The Geometrical Lectures of Isaac Barrow. Translated, with notes and proofs, by J. M. CHILD, B.A. (Cantab.), B.Sc. (Lond.). The Open Court Publishing Company, Chicago, 1916. xiv+218 pp.

"ISAAC BARROW was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673."

"The above is the ultimate conclusion that I have arrived at, as the result of six months' close study of a single book, my first essay in historical research. By the 'Infinitesimal Calculus,' I intend 'a complete set of standard forms for both the differential and integral sections of the subject, together with rules for their combination, such as for a product, a quotient, or a power of a function; and also a recognition and demonstration of the fact that differentiation and integration are inverse operations.'"

These are the opening paragraphs of the preface to this edition of Barrow's Lectures. While the rest of the book does perhaps not justify the claims of the preface, it furnishes a very welcome addition to the generally available information concerning Barrow. It presents, in abridged form, a translation by Mr. Child of the "Lectiones Geometricæ" of 1670, of which a first English translation was published by Edmond Stone in 1735. Numerous notes, bearing upon Mr. Child's thesis, are scattered throughout the text, proofs have been added in a number of places, and there is an introduction of

32 pages. There we find a brief account of the life and work of Barrow, of the work on infinitesimal calculus, done before his time and of the mutual influence of Barrow and Newton; a description of the original text, an argument to show that Barrow obtained his results analytically, although he published them in geometrical form, and finally a list of the analytical equivalents of his chief theorems.

Mr. Child appears to base his conclusions on the following points: (1) Barrow gave in his "Lectiones Geometricæ" a complete treatment of the elements of both the differential and integral calculus, including the fundamental theorem; (2) he obtained his results by means of some analytical device, essentially the same as that used by Newton; (3) the essential parts of the "Lectiones Geometricæ" were completed before 1664, the year in which "Barrow first came into close personal contact with Newton"; (4) Leibniz bought a copy of Barrow's work in 1673. The present reviewer believes that Mr. Child's work definitely establishes the first of these points and furnishes strong arguments for the second and third, without, however, settling them absolutely. Additional arguments for the conclusion concerning Leibniz are reserved for the future.

It seems not at all unlikely therefore that we shall have to place Barrow at least on a par with, if not above, Newton and Leibniz among the inventors of the calculus. Just where the Japanese mathematician Seki, whose "circle principle" antedates the work of Barrow, should be placed is a matter to be decided by historians who are concerned with questions of priority and with the influence of one mind upon another. To them I must also leave it to judge in how far Mr. Child's conclusions are justified by the evidence he has so far presented.

But entirely apart from their bearing upon historical questions, Barrow's lectures are of interest to the modern reader. In the first five lectures are developed various properties of curves, which are thought of kinematically, viz., as generated by a point descending with increasing velocity along a line which moves uniformly and parallel to itself. A tangent to a curve is spoken of as a line which has one point in common with the curve and which lies entirely on one side of it. In his fourth lecture, Barrow then shows that the slope of the tangent is equal to the slope of the curve at the point of tangency. He is thereby led to conceive of the tangent as "a prolongation of an element of the curve," which is "to be considered as composed of an infinite succession of infinitesimal straight lines."

The remaining seven lectures are devoted to a purely geometrical treatment of problems of tangents and areas, not, as was the case in the work of most of Barrow's predecessors, for special curves, but for general classes of curves. One of the characteristics of Barrow's method, and a consequence of its geometrical character, is the constant use of what we now know in our textbooks as the "subtangent" and "polar subtangent," and it is somewhat of a surprise to find these good friends outside a textbook. So we have in Lecture VIII the theorem, that if $\varphi(x) = a/f(x)$, then the subtangents for points with the same abscissa on the curves $y = f(x)$ and $y = \varphi(x)$ are equal but opposite in sign. In Lecture IX it is shown that if $[\varphi(x)]^m = a^{m-n}[f(x)]^n$, then the subtangents for corresponding points on the curves $y = \varphi(x)$ and $y = f(x)$ are in the ratio $m : n$; i. e., Barrow derives a formula for the differentiation of a fractional power. Each theorem on curves in cartesian coordinates is followed by a corresponding one dealing with curves given in polar coordinates. There are given formulas for the differentiation of the sum, the product and the quotient of two functions, for the differentiation and integration of tan θ, of $\sqrt{x^2 + a^2}$, etc. Let us close by quoting two theorems, appearing in Lectures X and XI respectively and constituting together the essence of the fundamental theorem of the calculus: "Let ZGE be any curve of which the axis is AD; and let ordinates applied to this axis, AZ, PG, DE, continually increase from the initial ordinate AZ; also let AIF be a line such that, if any straight line EDF is drawn perpendicular to AD, cutting the curves in the points E, F, and AD in D, the rectangle contained by DF and a given length R is equal to the intercepted space $ADEZ$; also let $DE : DF = R : DT$, and join DT. Then TF will touch the curve AIF"; and "Again, let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, $TF : FM = R : FZ$. Then the space $ADLK$ is equal to the rectangle contained by R and DB."

ARNOLD DRESDEN.

Die mathematischen Grundlagen der Variations- und Vererbungslehre. Von P. RIEBESELL. Leipzig, B. G. Teubner, 1916. 45 pp.

THIS little book is volume 34 of the Mathematische Bibliothek, published under the editorship of Lietzmann and Witting. It is doubtless intended for the biological, rather than for the mathematical public, although its title might lead one to expect a somewhat more fundamental analysis of the larger mathematical questions which arise in modern biological research. The first 32 pages are devoted to an exposition of the elements of the theory of probabilities and of the problems connected with the determination of a function that will represent a given distribution of statistical data. To make the task of reading the book as easy as possible for the non-mathematical reader, the theory is followed step by step by numerical examples. In the last 9 pages there is found a discussion of some of the biological questions in which the mathematical problems play a rôle. Probably the greatest use that one may expect from a book like this would come if it should stimulate some mathematician to acquaint himself further with the mathematical problems that arise in biological theory or if it should lead a biologist to more extended mathematical studies. If two sciences are to cooperate effectively, there must be trained "agents de liaison," who will make possible a partial overlapping of the spheres of interest.

ARNOLD DRESDEN.

Methoden zur Lösung geometrischer Aufgaben. Von B. KERST. (Mathematische Bibliothek herausgegeben von W. Lietzmann und A. Witting, Band 26.) Leipzig, Teubner, 1916. 12mo. 2 + 47 pages. Paper, price M.0.80.

AMONG the scores of books, pamphlets and articles which have been written concerning methods of solving problems of elementary synthetic geometry, the work of the late Julius Petersen, the Dane, is easily the best by reason of its elegant exposition, comprehensiveness, and suggestiveness. The first edition was published in 1866;* but the enlarged second edition, of which English and German translations were published in the same year, 1879, is the one whose contents are most familiar.†

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* Methoder og Theorier til Løsning af geometriske Konstruktionsopgaver, anvendte paa c. 300 Opgaver. Kjøbenhavn, 1866. 4 + 85 pp. + 1 pl.

† There have been at least 18 editions of this remarkable work: 6 in

In his "Vorwort" Kerst states: "Von der benutzten Literatur ist zu erwähnen: Tropfke, Geschichte der Elementar-Mathematik, 1903; Petersen, Methoden und Theorien zur Auflösung geometrischer Konstruktionsaufgaben, Kopenhagen, 1879 (im Buchhandel nicht mehr zu haben); R. Von Fischer-Benzon, Programm Kiel Gymnasium, 1884. Einige Aufgaben entstammen dem Aufgaben-Repertorium aus Hoffmanns Zeitschrift für math. u. naturw. Unterricht."

When it is recalled that Tropfke's work is wholly historical and that the part of Fischer-Benzon's work dealing with "method" is acknowledged to be largely due to Petersen, we would expect to find here much which has been made familiar by our old friend. In this expectation we are not disappointed. All of the methods and most of the problems are in Petersen's work.

It is, then, hardly necessary to indicate at length the nature of the contents of Kerst's little book except to say that it contains 17 paragraphs each developing a certain method, for the most part by the solution of problems. In all, there are seventy of the problems given with solutions, and about as many more to solve.

In presenting a solution of the problem of Apollonius by means of Petersen's parallel translation of circles, Kerst makes the unwarranted statement (page 47): "Das ist wahrscheinlich auch der Weg, auf dem Apollonius selbst zur Lösung gelangte."

There is probably no English reader in this country who would ever turn to this work in preference to that by Petersen. But of course the latter costs six times as much. It is doubtless significant of the dearth, or cost, of German supplies about two years ago that this number of the Mathematische Bibliothek appeared in paper covers instead of with the cloth back and board covers which had become familiar to us in connection with the first twenty volumes of the series.

R. C. ARCHIBALD.

Danish, 5 in French, 2 in Russian and one each in Dutch, English, German, Hungarian, and Italian. The English edition is readily obtainable from Copenhagen, and the French edition from Paris.

NOTES.

AT the meeting of the London mathematical society held March 14, 1918, the following papers were read: By G. H. HARDY, "The representation of a number as the sum of any number of squares"; by G. N. WATSON, "A problem in the theory of numbers."

THE following university courses in mathematics are announced:

CORNELL UNIVERSITY (academic year 1918–1919).—By Professor JAMES MCMAHON: Theory of probabilities, three hours; Introduction to actuarial science, three hours.—By Professor J. H. TANNER: Introduction to the mathematics of finance, two hours.—By Professor VIRGIL SNYDER: Descriptive geometry, three hours (first term); Analytic geometry of space, three hours (second term).—By Professor F. R. SHARPE: Hydrodynamics, three hours (first term); Elasticity, three hours (second term).—By Professor W. B. CARVER: Projective geometry, three hours.—By Professor ARTHUR RANUM: Line geometry, three hours (second term).—By Professor D. C. GILLESPIE: Differential equations, three hours.—By Professor W. A. HURWITZ: Differential equations of mathematical physics, three hours.—By Professor C. F. CRAIG: Functions of a complex variable, three hours.—By Professor F. W. OWENS: Advanced calculus, three hours.—By Dr. M. G. GABA: Problems in mathematics, three hours.

JOHNS HOPKINS UNIVERSITY (academic year 1918–1919).—By Professor FRANK MORLEY: Higher geometry, three hours (first term); Theory of functions, three hours (second term); Dynamics and hydrodynamics, two hours (second term).—By Professor A. B. COBLE: Theory of correspondences, two hours.—By Professor ABRAHAM COHEN: Elementary theory of functions, two hours; Applied mathematics, two hours (second term).

UNIVERSITY OF CALIFORNIA (summer session, June 24–August 3).—In addition to the courses offered at Los Angeles (see BULLETIN, this volume, page 363), the following advanced courses are to be given at Berkeley: By Professor D. N. LEHMER: Theory of numbers, five hours.—By Professor P. J. DANIELL (of the Rice Institute): Differential equations, five hours; Theory of integral equations, five hours.—By Dr. B. A. BERNSTEIN: Functions of a complex variable, five hours.

UNIVERSITY OF CHICAGO. Autumn quarter:—By Professor E. H. MOORE: Matrices in general analysis, three hours.—By Professor L. E. DICKSON: Theory of numbers, three hours. —By Professor E. J. WILCZYNSKI: Projective differential geometry, three hours.—By Professor H. E. SLAUGHT: Differential equations, three hours.—By Professor A. C. LUNN: Heat and molecular physics, three hours; Electron theory, three hours.—Winter quarter:—By Professor E. H. MOORE: Functions of infinitely many variables, three hours.—By Professor L. E. DICKSON: Algebraic numbers, three hours.—By Professor G. A. BLISS: Definite integrals, three hours; Differential equations, three hours.—By Professor E. J. WILCZYNSKI: Projective differential geometry, II, three hours. —By Professor A. C. LUNN: Thermodynamics, three hours; Theory of sound, three hours.—Spring quarter:—By Professor E. H. MOORE: Functions of infinitely many variables, II, three hours; Limits and series, three hours.—By Professor L. E. DICKSON: Linear algebra, three hours; Solid analytics, three hours.—By Professor G. A. BLISS: Functions of lines, three hours; Partial differential equations, three hours; Functions of a complex variable, three hours.—By Professor A. C. LUNN: Geometric optics, three hours.

UNIVERSITY OF ILLINOIS (academic year 1918–1919).—By Professor E. J. TOWNSEND: Functions of a complex variable, three hours; Differential equations and advanced calculus, three hours.—By Professor G. A. MILLER: Continuous groups, three hours (second term); Theory of equations, three hours (first term).—By Professor H. L. RIETZ: Actuarial theory, three hours.—By Professor J. STEBBINS: Least squares, two hours (first term).—By Professor J. B. SHAW: Fundamental functions, three hours (first term); Functional transformations, three hours (second term).—By Professor C. H. SISAM: Invariants and higher plane curves, three hours; Solid analytic geometry, three hours (second term).—By Professor R. D. CARMICHAEL: Elliptic functions, three hours.

—By Professor ARNOLD EMCH: Projective geometry, three hours.—By Professor A. R. CRATHORNE: Calculus of variations, three hours.—By Dr. E. B. LYTLE: History of mathematics, two hours (second term).—By Dr. G. A. WAHLIN: Theory of numbers, three hours.

PROFESSOR GABRIEL KOENIGS has been elected member of the Paris academy of sciences.

PROFESSOR G. CASTELNUOVO, of the University of Rome, has been elected national member of the Reale Accademia dei Lincei; Professors E. BOREL and E. GOURSAT, of the University of Paris, Professor J. HADAMARD, of the Collège de France, and Professor H. LAMB, of the University of Manchester, have been elected foreign associates.

THE Turin academy has elected the following corresponding members: Professors L. BERZOLARI, of the University of Pavia; R. MARCOLONGO, of the University of Naples; L. PINCHERLE, of the University of Bologna; G. RICCI and F. SEVERI, of the University of Padua; G. ALBENGA, G. COLOMETTI and G. A. MAGGI, of the University of Pisa; V. REINA, of the University of Rome.

DR. E. G. TOGLIATTI has been appointed docent in projective and descriptive geometry at the University of Turin.

DR. A. VERGERIO has been appointed docent in analysis at the University of Bologna.

DR. E. G. BILL, assistant professor of mathematics at Dartmouth College, has been granted leave of absence to serve on the staff of the Military Service Council at Ottawa, Canada.

AT the University of California, Professor FLORIAN CAJORI, of Colorado College, has been appointed professor of the history of mathematics. Associate professor D. N. LEHMER has been promoted to a full professorship of mathematics. Dr. B. A. BERNSTEIN, Dr. THOMAS BUCK, and Dr. FRANK IRWIN have been promoted to assistant professorships Professor G. C. EDWARDS has been made professor emeritus, after forty-five years of service.

ASSISTANT professor ANNA J. PELL, of Mount Holyoke College, has been appointed associate professor of mathematics at Bryn Mawr College. Dr. OLIVE C. HAZLETT, of Bryn Mawr, has been appointed assistant professor of mathematics at Mount Holyoke.

ASSISTANT professor H. M. SHOWMAN, of the Colorado School of Mines, has been promoted to a professorship of mechanics and engineering.

MR. H. L. OLSON has been appointed instructor in mathematics at Heidelberg University, Tiffin, Ohio.

DR. J. V. DEPORTE, of the New York State College at Albany, has been promoted to an assistant professorship of mathematics.

PROFESSOR E. E. LEVI, of the University of Genoa, was killed in battle October 28, 1917, at the age of 34 years. The Italian society of sciences (the XL) awarded him its gold medal in 1912.

PROFESSOR A. VITERBI, of the University of Pavia, was killed in battle November 18, 1917, at the age of 44 years.

BOOK CATALOGUES:—Galloway and Porter, Cambridge, England, list of 78 titles, including 15 in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. W. R.). Cambridge papers. London, Macmillan, 1918. 8vo. 6+326 pp. 6s.

BERICHTE und Mitteilungen veranlasst durch die Internationale Mathematische Unterrichtskommission. 2te Folge, III: Titel und Inhaltsubersicht zur ersten und zweiten Folge der Berichte und Mitteilungen. Leipzig, Teubner, 1917. 8vo. 16+99 pp. M. 4.00

GAUJA (P.). Les fondations de l'Académie des Sciences (1881–1915), avec une préface par A. Lacroix et E. Picard, secrétaires perpetuels de l'Académie des Sciences. Paris, Gauthier-Villars, 1917. 8vo. 6+658 pp. Fr. 12.00

Koenigsberger (L.). Weierstrass' erste Vorlesung uber die Theorie der elliptischen Funktionen. Leipzig, Teubner, 1917. 8vo. 32 pp.

Lacroix (A.). See Gauja (P.).

Picard (E.). See Gauja (P.).

Viscardini (M.). Nota su alcune corrispondenze duali di carattere proiettivo, metrico e statico sulla sfera e sul piano. Roma, Stabilimento tipo-litografico del Genio Civile, 1918. 12 pp. 2 tavole.

II. ELEMENTARY MATHEMATICS.

Anderson (R. F.). See Hallett (G. H.).

Bassi (A.). Esercizi e problemi di algebra complementare. 2a edizione riveduta. Volume 1 (per la terza classe degli istituti tecnici). Parte 2. Livorno, Giusti, 1918. 32mo. 8+210 pp. L. 2.10

Boston School Committee. Outline of work in mathematics for intermediate classes in elementary grades. (School documents No. 12.) Boston, School Committee, 1916. 8vo. 9 pp.

Hallett (G. H.) and Anderson (R. F.). Elementary algebra. Boston, Silver-Burdett, 1917. 8vo. 6+402 pp.

Keal (H. M.) and Phelps (N. S.). Eighth grade mathematics. Chicago, Atkinson, Mentzer and Company, 1917. 8vo. 5+132 pp. $0.65

——. Secondary mathematics. Book 1. Chicago, Atkinson, Mentzer and Company, 1917. 8vo. 224 pp. $0.80

Mendizabal Tamborrel (J. de). Tratado de goniometria. 2a edicion. Mexico, Departamento de Talleres Graficos de la Secretaria de Fomento, 1917. 8vo. 209 pp.

Milne (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College for the years 1908–1917. Edited by R. M. Milne. London, Macmillan, 1918. 8vo. 7s.

Myers (G. W.). See Palmer (C. I.).

Palmer (C. I.) and Taylor (D. P.). Teachers' handbook for Palmer and Taylor's plane geometry; edited by G. W. Myers. (Lake Mathematical Series.) Chicago, Scott, Foresman and Company, 1917. 8vo. 14+232 pp. $2.50

Phelps (N. S.). See Keal (H. M.).

Taylor (D. P.). See Palmer (C. I.).

III. APPLIED MATHEMATICS.

Annuaire pour l'an 1918 publié par le Bureau des Longitudes. Avec des notices scientifiques par G. Bigourdan, J. Renaud, M. Hamy et E. Picard. Paris, Gauthier-Villars, 1918. 16mo. 10+870 pp. Fr. 2.00

Amagat (E. H.) et Décombe (L.). La statique des fluides, la liquéfaction des gaz et l'industrie du froid. 1re et 2e partie. Paris et Liège, Béranger, 1917. 6+265 pp. Fr. 18.00

Aughtie (H.). Applied mechanics, second year. London, Routledge, 1918. 8vo. 2s. 6d.

Bailey (W. B.) and Cummings (J.). Statistics. Chicago, McClurg, 1917. 6+153 pp. Cloth. $0.60

Bigourdan (G.). See Annuaire.

Boston School Committee. Special syllabus, drawing and manual training; grades 1–8, 1917–1918. (School documents Nos. 13–14.) 2 volumes. Boston, School Committee, 1917. 8vo.

Cousin (A.). Résumé pratique de navigation. Long cours. Plaisance. Cabotage. 2e édition. Paris, Challamel, 1917. 8vo. Cartonné. Fr. 6.50

Cummings (J.). See Bailey (W. B.).

Decante (E.). Tables d'azimut pour tous les points situés entre les cercles polaires et les astres dont la déclination est compris entre 0° et 48°. 3e édition augmentée. Tome 8: 59° à 66°. Paris, Gauthier-Villars, 1917. 8vo. Fr. 6.00

Décombe (L.). See Armagat (E. H.).

Dunwoody (H.). Mechanics, sound, light, thermo-mechanics and hydraulics. London, Chapman and Hall, 1918. 8vo. 13s. 6d.

Elderton (W. P.). Frequency curves and correlation. Addendum with diagram and errata. London, Layton, 1917. 22 pp. 1s. 6d.

Hamy (M.). See Annuaire.

Hartman (W. B.). Machine shop practice. New York, Appleton, 1917. 10+247 pp. Cloth.

Hosmer (G. L.). Navigation. New York, Wiley, 1918. 12 mo. 9+214 pp. Cloth. $1.25

Hughes (C. H.). Handbook of ship calculations, construction and operation. A book of reference for ship owners, ship officers, ship and engine draughtsmen, marine engineers and others engaged in the building and operating of ships. New York, Appleton, 1917. 24+740 pp. Leather.

Knibbs (G. H.). The mathematical theory of population applied to the data of Australian population statistics. Appendix A. Volume 1. Melbourne, McCarron, Bird and Company, 1917. 16+466 pp.

Moritz (F.). Méthode de transformation des diagrammes de Watt (volume et pression) en diagrammes entropiques par le réseau entropique. Paris, Gauthier-Villars, 1917. 16mo. 10 pp. Fr. 1.80

Muir (W. C. P.). A treatise on navigation and nautical astronomy including the theory of compass deviations prepared for use as a text-book at the U. S. Naval Academy. Fourth edition revised and enlarged. Annapolis, Md., The United States Naval Institute, 1918. 18+784 pp. $4.20

d'Ocagne (M.). Principes de la nomographie. Application des nomogrammes à l'alignement aux différents cas de résolution des triangles sphériques. Paris, Gauthier-Villars, 1917. 8vo. 42 pp. Fr. 1.50

Perry (H.). Theories of energy. New York, Putnam, 1918. 8+231 pp. Cloth. $1.75

Picard (E.). See Annuaire.

Renaud (J.). See Annuaire.

Thomas (A. O.). Rural arithmetic. New York, American Book Company, 1918.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and ninety-eighth regular meeting of the Society was held in New York City on Saturday, April 27, 1918, extending through the usual morning and afternoon sessions. The attendance included the following thirty-three members:

Professor R. C. Archibald, Dr. F. W. Beal, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Professor D. R. Curtiss, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Dr. Olive C. Hazlett, Dr. T. R. Hollcroft, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. R. Kline, Professor P. H. Linehan, Professor R. B. McClenon, Professor L. C. Mathewson, Professor R. L. Moore, Professor F. M. Morgan, Professor Anna J. Pell, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor H. S. White, Mr. J. K. Whittemore.

Ex-President H. S. White presided at the morning session and Professor W. B. Fite at the afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. O. S. Adams, U. S. Coast Survey; Professor W. P. Parker, Union Christian College, Pyeng Yang, Korea; Dr. E. F. Simonds, University of Illinois. Seven applications for membership in the Society were received.

Professor P. F. Smith was reelected a member of the Editorial Committee of the *Transactions*. Committees were appointed to consider the question of the publication of the recent Chicago Symposium, and to submit to the Council at the October meeting a list of nominations of officers to be elected at the annual meeting.

The following papers were read at the April meeting:

(1) Professor ARNOLD EMCH: "On plane algebraic curves with a given system of foci."

(2) Dr. J. F. RITT: "On the iteration of polynomials."

(3) Professor F. F. DECKER: "On the order of the system of equations arising from the vanishing of determinants of a given matrix."

(4) Professor O. E. GLENN: "Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic."

(5) Professor J. E. ROWE: "The quinquesecant line invariant of the rational sextic curve in space."

(6) Professor F. H. SAFFORD: "Parametric equations of the path of a projectile when the air resistance varies as the nth power of the velocity."

(7) Professor C. L. E. MOORE: "Surfaces of rotation in space of four dimensions."

(8) Professor C. L. E. MOORE: "Translation surfaces in hyperspace."

(9) Dr. MARY F. CURTIS: "Note on the rectifiability of a space cubic."

(10) Professors F. R. SHARPE and VIRGIL SNYDER: "Certain types of involutorial space transformations."

(11) Dr. CAROLINE E. SEELY: "On kernels of positive type."

(12) Lieut. J. W. HOPKINS: "Some convergent developments associated with irregular boundary conditions."

(13) Dr. J. R. KLINE: "A necessary and sufficient condition that a closed connected point set that divides the plane into two domains be a simple curve."

(14) Professor EDWARD KASNER: "Equilong symmetries and a related group."

(15) Professor H. B. PHILLIPS: "Functions of matrices."

(16) Mr. G. H. HALLETT, JR.: "Linear order in three-dimensional euclidean and double elliptic spaces."

(17) Mr. H. S. VANDIVER: "On transformations of the Kummer criteria in connection with Fermat's last theorem."

(18) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem (third paper)."

(19) Mr. H. S. VANDIVER: "On the first factor of the class number of a cyclotomic field."

(20) Mr. H. S. VANDIVER: "Proof of a property of the norm of a cyclotomic integer."

Mr. Hopkins's paper was communicated to the Society and read by Professor Dunham Jackson. Mr. Hallett was introduced by Professor R. L. Moore. In the absence of the authors, the papers of Professor Emch, Professor Decker, Professor Rowe, Professor Safford, Professor C. L. E. Moore, Dr. Curtis, Professors Sharpe and Snyder, Professor Phillips, and the last two papers of Mr. Vandiver were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The methods of Plücker, Siebeck, and others for determining the foci of given algebraic curves are well known. In this paper Professor Emch assumes conversely a given set of foci and determines the system of algebraic curves with the given set of foci. As an application of the method, among other problems, the equation

$$X^{n-2k}\, y^{2k} = (-1)^{n-2k} \frac{(2k)^{2k}}{n^n\,(n-2k)^{2k-n}}$$

of a special class of n-ics, with the nth roots of unity interpreted in the cartesian plane as foci, is established.

2. Let $f(x)$ be a polynomial of degree n greater than unity, and $f_p(x)$ its pth iterate. If $|x|$ is sufficiently large, $f_p(x)$ goes to infinity with p. Dr. Ritt studies the domain which is drawn to infinity. This study is made by means of a function $\varphi(x)$, introduced by L. Boettcher, which is defined by the relation

$$\varphi(x) = \lim_{p=\infty} [f_p(x)]^{1/n^p}.$$

It follows from the existence of this function that the domain drawn to infinity is a connected region. If none of the roots of $f'(x) = 0$ lies in this domain, the domain is simply connected. If the coefficient of x^n in $f(x)$ is unity, this simply connected domain covers a portion of the interior of the circle $|x| = 1$. If a root of $f'(x) = 0$ lies in the domain drawn to infinity, that domain is infinitely connected.

3. In a paper previously presented to the Society Professor Decker gave a proof of a theorem announced by Salmon without proof: The number of solutions of the system of equations arising from the vanishing of all determinants of the mth order that can be formed from a matrix with m rows and n columns, $m \ngtr n$, by the suppression of $n - m$ columns, the element in the ith row and jth column being of degree $\alpha_i + \alpha_j$

in $n - m + 1$ non-homogeneous variables, is K_{n-m+1}, where

$$K_l = \sum_{i=0}^{i=l} c_i \, \delta_{l-i},$$

c_l representing the sum all possible products of l different α's and δ_l the sum of all possible products of l α's, repetitions being permissible.

In the present paper the system arising from the vanishing of all determinants of the rth order that can be formed by the suppression of $n - r$ columns and $m - r$ rows, $r \not> m$, $r \not> n$, is treated. The order is shown to be

$$\begin{vmatrix} K_{n-r+1} & K_{n-r} & \cdots & K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \cdots & K_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n-2r+m+1} & K_{n-2r+m} & \cdots & K_{n-r+1} \end{vmatrix}.$$

If the degree of every element is b, the order reduces to

$$\frac{\prod\limits_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1} \cdot b^{(m-r+1)(n-r+1)}}{\prod\limits_{j=1}^{j=m-r+1} {}_{n-r+j}C_{n-r+1}},$$

a result established by Segre.

4. Professor Glenn's paper is devoted primarily to a principle in modular invariant theory based, in a general theoretical way, upon analogy with the algebraical principle of symbolical convolution, by which, in both theories, an assigned covariant of a given degree becomes a member of a finite scale of derived concomitants of the same degree. Upon this principle the main theorems in algebraical invariant theory have been made to rest, and, in the present paper, the elements of a similar general modular theory are considered.

The methods developed are applied in deriving a complete system of formal concomitants of the binary quadratic form under the total binary group modulo 3.

The paper will be offered for publication in the *Transactions*.

5. In Professor Rowe's paper the $u_t^6 = 0$, $v_t^6 = 0$, and $w_t^6 = 0$ are three linearly independent binary sextics of the

double infinity of binary sextics to which all plane sections of the R_3^6, the rational sextic curve in ordinary space, are apolar, and $a_t^5 = 0$ is a binary quintic. The necessary and sufficient condition that the R_3^6 have a quinquesecant line is that $|\alpha u|^5 u_t = 0$, $|\alpha v|^5 v_t = 0$, and $|\alpha w|^5 w_t = 0$ hold for all values of t. This gives rise to an invariant of degree two in the four-rowed determinants of the matrix of the coefficients of the parametric equations of R_3^6. By a simple substitution this invariant may be transformed into an invariant of the rational plane sextic.

6. Professor Safford has obtained in parametric form the equations of the path of a projectile when the air resistance varies as the nth power of the velocity of the projectile. This problem was proposed by a member of the National Research Council and it was stated that the solution was known only for the case of $n = 2$, as given by de Sparre in the *Comptes Rendus*, volume 160. In that case certain approximations were made in the early stages of the discussion. In the present paper the accuracy of the results depends only upon the accuracy with which the physical constants can be determined.

7. Professor Moore defines a surface of revolution as one left invariant by a rotation and discusses the forms of such surfaces if the rotation is restricted to the type having an infinite number of invariant planes.

8. In this paper Professor Moore shows that no translation surfaces in hyperspace, except cylinders, can be expressed in more than one way as translation surfaces. The condition that the surface be developable is also discussed.

9. A necessary and sufficient condition that the space cubic

$$x_1 = at, \qquad x_2 = bt^2, \qquad x_3 = ct^3 \qquad (abc \neq 0)$$

be algebraically rectifiable is, as Dr. Curtis shows, that the cubic be a helix.

10. A (2, 1) correspondence between the points of two spaces (x) and (x') may be defined by three equations which are linear in (x'). The transformation which interchanges the two points in (x) which correspond to any point in (x') is

involutorial. By this method, Professors Sharpe and Snyder discuss various involutorial transformations, in particular the case in which the equations represent quartic surfaces through an octic curve of genus two. Each quartic surface is invariant under the transformation which is of order 31, and a similar transformation. The product of the two involutions is of infinite order, so that the surfaces are similar to but distinct from the known Fano quartic surfaces through a sextic of genus two.

11. Dr. Seely considers kernels that have an infinite number of iterated kernels of positive type with respect to all functions of the form $\alpha\varphi_i(s) + \beta\varphi_j(s)$ and $\alpha\psi_i(s) + \beta\psi_j(s)$, where the φ's and ψ's are characteristic functions, and shows that such kernels have certain properties in common with symmetric kernels.

12. Birkhoff (*Transactions*, 1908) has given a general theory of the convergence of the development of an arbitrary function $f(x)$ in series of characteristic solutions of an ordinary linear differential equation of the nth order with "regular" boundary conditions. It was found by Jackson (*Proceedings of the American Academy*, 1915) that, for an extensive class of irregular boundary conditions, hypotheses similar to those imposed by Birkhoff on $f(x)$ are very far from being sufficient to insure convergence of the corresponding expansion. Lieutenant Hopkins examines more carefully the case of the differential equation $d^3u/dx^3 + \rho^3 u = 0$, with the irregular boundary conditions $u(0) = u'(0) = u(\pi) = 0$. He finds that if a function $f(x)$ is to have a uniformly convergent development in terms of the characteristic solutions of this system, it must be analytic throughout a triangular region surrounding the origin in the complex x-plane, and must furthermore be of the particular form $x^2\varphi(x^3)$, where φ is an analytic function of its argument. That is, $f''(x)$ must be an analytic function of x, invariant under the transformation $x' = \omega x$, where ω is a complex cube root of unity. He proves conversely that if $f(x)$ does have the properties indicated, throughout a suitable region, its development will converge uniformly to the value $f(x)$ throughout any interval contained in $(0, \pi)$ and not reaching out to the terminal point $x = \pi$. He has thereby made an essential contribution to the theory of a class of series

which have some of the properties of power series on the one hand and of ordinary trigonometric series on the other, and which have further novel and striking properties of their own.

13. Dr. Kline proves the following theorem:

A necessary and sufficient condition that a closed, connected, connected in kleinem, plane point set should be a simple curve is that it divide its plane into two mutually exclusive domains.

14. The comparison of conformal geometry and equilong geometry brings out many fundamental differences as well as many analogies. In the present paper Professor Kasner defines the particular equilong transformations which play the same rôle in the equilong theory as Schwarzian reflections or conformal symmetries in the theory of functions of a complex variable. The analytic representation as well as the geometric interpretation is much simpler in the equilong theory. The group generated by symmetries is also obtained.

15. In the paper of Professor Phillips a canonical form is given for a function of two or more commutative matrices, analogous to Sylvester's formula for a function of a single matrix. This is used to represent analytic functions of two or more matrices. If z is a complex variable and Z and A matrices, it is shown that $f(Z)$ can be expanded in a Taylor's series in powers of $Z-A$, the series converging if each root of Z lies within a circle, with center at the corresponding root of A, in which $f(z)$ is analytic. A simple example is given of two commutative matrices not expressible as polynomials in the same matrix.

16. Mr. Hallett gives four pairs of independent categorical postulate sets for euclidean and double elliptic geometries of three dimensions in terms of point and order. Each pair has a large common basis. Each set is such that all the properties of a line can be deduced from it without the use of any postulates which necessitate the existence of two dimensions. In three pairs of sets a modification of order due to Kempe is used to advantage. A fifth treatment, involving fewer postulates than the others, is given for double elliptic geometry. The categoricity of the sets is proved by establishing the equivalence of sets of postulates given previously by Professor Veblen and Dr. Kline.

17. In an article in the Berlin *Sitzungsberichte*, June, 1914, Frobenius proved that if

$$(1) \qquad x^p + y^p + z^p = 0$$

is satisfied in integers prime to the prime p, then $11^{p-1} \equiv 17^{p-1} \equiv 1 \pmod{p^2}$ and if $p \equiv 2 \pmod 3$ also $7^{p-1} \equiv 13^{p-1} \equiv 19^{p-1} \equiv 1 \pmod{p^2}$. He also stated (loc. cit. page 670) that it is impossible to derive the congruence $23^{p-1} \equiv 1 \pmod{p^2}$ as a criterion for (1) by the use of his methods for the case $p \equiv 1 \pmod 3$ because of the appearance of a factor $t^2 - t + 1$ in the process of eliminating certain functions t. This statement of Frobenius is incorrect. In the present paper Mr. Vandiver derives the condition $23^{p-1} \equiv 1 \pmod{p^2}$ for the solution of (1) in integers prime to p, provided p is not of the form 1 (mod 11), by using only relations given by Frobenius in the article cited above.

18. In previous papers presented to the Society under this title in January and April, 1915, Mr. Vandiver derived a certain relation involving cyclotomic integers which serves as a starting point for the derivation of a number of criteria relating to Fermat's last theorem. In the present paper the criteria $B_s \equiv 0 \pmod{p^2}$

$$s = \frac{(p - 2i)\,p + 1}{2} \qquad (i = 2, 3, 4, 5),$$

for the solution of equation (1) of the preceding abstract in integers prime to p are obtained, the B's being numbers of Bernoulli, $B_1 = 1/6$, $B_2 = 1/30$, etc. These criteria reduce, modulo p, to the conditions

$$B_{\frac{1}{2}(p-3)} \equiv B_{\frac{1}{2}(p-5)} \equiv B_{\frac{1}{2}(p-7)} \equiv B_{\frac{1}{2}(p-9)} \equiv 0 \pmod{p}$$

given by Kummer and Mirimanoff.

19. The number of classes of ideals in a cyclotomic field defined by $e^{2i\pi/p}$, where p is prime, may be expressed as the product of two integral factors, one of which (generally known as the first factor) is as follows:

$$h = \frac{f(Z)\,f(Z^3)\ \ldots\ f(Z^{p-2})}{(2p)^{\frac{1}{2}(p-3)}}$$

where

$$f(x) = r_0 + r_1 x + \ldots + r_{p-2} x^{p-2},$$

$Z = e^{2i\pi/(p-1)}$, and r_i is the least positive residue of r^i, modulo p. Mr. Vandiver derives necessary and sufficient conditions that h be divisible by p^n, in terms of Bernoulli numbers, the argument used being different from those employed by Kummer and Kronecker in their treatment of the case $n = 1$.

20. In his first proof of the law of reciprocity between two ideals in a regular cyclotomic field Kummer gave the relation*

$$\left[\frac{d^{p-1} \log \omega(e^v)}{dv^{p-1}}\right]_{v=0} \equiv \frac{(\omega(1))^{p-1} - N(\omega)}{p} \pmod{p},$$

where

$$\omega(x) = a_0 + a_1 x + \ldots + a_{p-2} x^{p-2},$$

$$\omega = a_0 + a_1 \alpha + \ldots + a_{p-2} \alpha^{p-2},$$

$\alpha = e^{2i\pi/p}$ and $a_0, a_1, \ldots, a_{p-2}$ are integers, ω being prime to the prime p. In the present paper Mr. Vandiver gives a proof of this result which differs in character from that of Kummer.

F. N. COLE,
Secretary.

NOTE ON ISOGENOUS COMPLEX FUNCTIONS OF CURVES.

BY PROFESSOR W. C. GRAUSTEIN.

(Read before the American Mathematical Society September 5, 1917.)

LET L be a continuous, closed and directed space curve, without multiple points, let $-L$ be the same curve oppositely directed, and let $F|[L]|$ be a function of L, such that $F|[-L]| = -F|[L]|$. Form the ratio

$$\frac{F|[L']| - F|[L]|}{\sigma},$$

where L' is the directed curve obtained from L by replacing an arc PP' of L by a second arc joining P with P', and σ

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* *Abhandlungen, Berlin Academy*, 1859, p. 119, formula (7).

is the area of a surface S bounded by the two arcs PP'. The limit of this ratio as the surface S shrinks to a point in P, if it exists, is called the derivative of $F\,|\,[L]\,|$ with respect to S in the point P and is denoted by $dF/d\sigma$. This derivative, dependent in general on L as well as on P and S, becomes for a given S a function of P alone, in the case that $F\,|\,[L]\,|$ is a so-called function of the first degree, that is, is a function such that the sum of its values for two directed curves, possessing a common arc oppositely directed for the two, is equal to its value for the curve resulting from the combination of the parts of the two curves other than their common arc. In this case the above ratio becomes $F\,|\,[L'']\,|\,/\sigma$, where L'' is the directed boundary of S, made up of the directed arcs PP' of L' and $-L$. Furthermore, in this case,

$$F\,|\,[L'']\,| = \int_S\!\int (p \cos nx + q \cos ny + r \cos nz)d\sigma,$$

where p, q, r are continuous point functions and $\cos nx$, $\cos ny$, $\cos nz$ are the direction cosines of the normal to S suitably directed. Then the value of the derivative of F with respect to S in the point P is

$$\frac{dF}{d\sigma} = \lim_{\sigma=0} \frac{F\,|\,[L'']\,|}{\sigma} = p \cos nx + q \cos ny + r \cos nz,$$

provided that the values of p, q, r, $\cos nx$, $\cos ny$, $\cos nz$ are taken in P.*

The functions p, q, r are termed the derivatives of F in P with respect to the three coordinate planes. Consider them as the components of a vector point function or vector c, and call this vector c the gradient of the function F in P. If the gradient c, besides being continuous, possesses continuous first partial derivatives, it follows, by application of the theorem of Gauss expressing a volume integral in terms of a surface integral, that

$$\operatorname{div} c = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z} = 0.$$

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* For a more detailed development of the contents of this and the two following paragraphs, see Volterra, "Sur une généralisation de la théorie des fonctions d'une variable imaginaire," *Acta Mathematica*, vol. 12 (1889), pp. 233–286; Lévy, "Sur les equations intégro-differentielles définissant des fonctions de lignes," Doctor's Thesis, Paris, 1911, Introduction; Evans, "Topics from the theory and applications of functionals, including integral equations," The Cambridge Colloquium Lectures, delivered September 6–8, 1916, Lectures I, II.

Conversely, if div $c = 0$, where c is a vector continuous with continuous first partial derivatives, there exists a function of curves, $F \mid [L] \mid$, of which c is the gradient.

Consider now

$$F = F_1 + iF_2, \quad \Phi = \Phi_1 + i\Phi_2,$$

where

$$F_1 = F_1 \mid [L] \mid, \quad F_2 = F_2 \mid [L] \mid, \quad \Phi_1 = \Phi_1 \mid [L] \mid, \quad \Phi_2 = \Phi_2 \mid [L] \mid$$

are functions of L of the first degree possessing gradients $c_1, c_2, \gamma_1, \gamma_2$, which are continuous with continuous first partial derivatives. Form the ratio $\Delta\Phi/\Delta F$, where ΔF and $\Delta\Phi$ are the variations in F and Φ due to replacing the arc PP' of L by a second arc joining P with P'. Evidently the limit, $d\Phi/dF$, of this ratio, as the surface S bounded by the two arcs PP' shrinks to a point in P, is $(d\Phi/d\sigma) : (dF/d\sigma)$. It depends, in general, not merely on P but on the aspect of the surface S in P, namely, on the values of cos nx, cos ny, cos nz in P. The condition necessary and sufficient that it be independent of S is that the vectors $c_1 + ic_2$, $\gamma_1 + i\gamma_2$ be proportional, or, in other words, that the vector product $\overline{c_1 + ic_2} \times \overline{\gamma_1 + i\gamma_2}$ vanish, that is, that

$$(1) \qquad c_1 \times \gamma_1 - c_2 \times \gamma_2 = 0, \quad c_1 \times \gamma_2 + c_2 \times \gamma_1 = 0.$$

If these conditions (1) are satisfied, $d\Phi/dF$ is dependent only on P, and the two complex functions $F = F_1 + iF_2$, $\Phi = \Phi_1 + i\Phi_2$ are said to be isogenous.

Lévy has noted,* without going into the details of hypothesis or proof, the general fact that for a given complex function $F = F_1 + iF_2$ there exist complex functions $\Phi = \Phi_1 + i\Phi_2$ isogenous to it, depending on two arbitrary functions of two independent parameters. In proving this, we assume that the functions F_1, F_2 possess gradients c_1, c_2, which are analytic, in general, in the coordinates x, y, z of the point P. If, then, we can establish the existence of analytic vectors γ_1, γ_2, satisfying the equations (1) and having divergences equal to zero, the existence of functions $\Phi = \Phi_1 + i\Phi_2$ isogenous to $F = F_1 + iF_2$ will follow.

The vectors γ_1, γ_2 will satisfy (1) when and only when they are linear combinations of the vectors c_1, c_2 of the form

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* Loc. cit., p. 11.

$$\gamma_1 = k_1 c_1 + k_2 c_2, \quad \gamma_2 = k_2 c_1 - k_1 c_2,$$

where k_1 and k_2 are scalar point functions. Thus our problem reduces to the determination of k_1, k_2 as solutions of the equations

$$\operatorname{div}(k_1 c_1 + k_2 c_2) = 0, \quad \operatorname{div}(k_2 c_1 - k_1 c_2) = 0,$$

or better, as solutions of the equations (2), to which these reduce by virtue of the vanishing of the divergences of c_1, c_2,

$$(2) \quad \begin{aligned} p_1 \frac{\partial k_1}{\partial x} + q_1 \frac{\partial k_1}{\partial y} + r_1 \frac{\partial k_1}{\partial z} + p_2 \frac{\partial k_2}{\partial x} + q_2 \frac{\partial k_2}{\partial y} + r_2 \frac{\partial k_2}{\partial z} &= 0, \\ p_2 \frac{\partial k_1}{\partial x} + q_2 \frac{\partial k_1}{\partial y} + r_2 \frac{\partial k_1}{\partial z} - p_1 \frac{\partial k_2}{\partial x} - q_1 \frac{\partial k_2}{\partial y} - r_1 \frac{\partial k_2}{\partial z} &= 0. \end{aligned}$$

But according to the theory of partial differential equations there exist simultaneous solutions k_1, k_2 of the equations (2) which are analytic in general and depend on two arbitrary functions of two independent variables.

To go more into detail concerning these solutions, let us consider instead of the equations (2) the equivalent equations (3), obtained by solving the equations (2) for $\partial k_1/\partial z$, $\partial k_2/\partial z$,

$$(3) \quad \begin{aligned} (r_1^2 + r_2^2) \frac{\partial k_1}{\partial z} + (p_1 r_1 + p_2 r_2) \frac{\partial k_1}{\partial x} + (q_1 r_1 + q_2 r_2) \frac{\partial k_1}{\partial y} & \\ + (p_2 r_1 - p_1 r_2) \frac{\partial k_2}{\partial x} + (q_2 r_1 - q_1 r_2) \frac{\partial k_2}{\partial y} &= 0, \\ (r_1^2 + r_2^2) \frac{\partial k_2}{\partial z} + (p_1 r_2 - p_2 r_1) \frac{\partial k_1}{\partial x} + (q_1 r_2 - q_2 r_1) \frac{\partial k_1}{\partial y} & \\ + (p_1 r_1 + p_2 r_2) \frac{\partial k_2}{\partial x} + (q_1 r_1 + q_2 r_2) \frac{\partial k_2}{\partial y} &= 0. \end{aligned}$$

There exist unique solutions k_1, k_2 of the equations (3), analytic in the neighborhood of (x_0, y_0, z_0), taking for $z = z_0$ given values $m_1(x, y)$, $m_2(x, y)$, and possessing in (x_0, y_0, z_0) given partial derivatives $(\partial k_1/\partial x)_0$, $(\partial k_1/\partial y)_0$, $(\partial k_2/\partial x)_0$, $(\partial k_2/\partial y)_0$, provided that the functions to which $\partial k_1/\partial z$ and $\partial k_2/\partial z$ are actually equal in equations (3) are themselves analytic in the neighborhood of $(x_0, y_0, z_0, (\partial k_1/\partial x)_0, (\partial k_1/\partial y)_0, (\partial k_2/\partial x)_0, (\partial k_2/\partial y)_0)$,* that is to say, provided that the vectors c_1, c_2 are analytic in (x_0, y_0, z_0) and r_1 and r_2 are not both zero there. These solutions of (3) are, then, analytic except perhaps in points of singularity of c_1, c_2 and in points for which $r_1 = r_2 = 0$. But they are identical with certain solutions of the equations (2), solved for $\partial k_1/\partial y$, $\partial k_2/\partial y$,—solutions which are analytic except perhaps in points of singularity of c_1, c_2 and in points for which $q_1 = q_2 = 0$. Evidently, then, these solutions k_1, k_2, and hence the vectors $\gamma_1 = k_1c_1 + k_2c_2$, $\gamma_2 = k_2c_1 - k_1c_2$, are analytic except perhaps in points of singularity of c_1, c_2 and in points in which both c_1 and c_2 are indeterminate, that is, have all three components zero. Thus we have the theorem:

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* Cf., e. g., Goursat, Cours d'Analyse mathématique, vol. 2, p. 363.

If the gradients c_1, c_2 of the functions F_1, F_2 in $F = F_1 + iF_2$, where $F_1 = F_1 | [L] |$, $F_2 = F_2 | [L] |$ are functions of the first degree of the space curve L, are in general analytic, the gradients γ_1, γ_2 of Φ_1, Φ_2 in $\Phi = \Phi_1 + i\Phi_2$, an arbitrary complex function of L of the first degree isogenous to F, are analytic save perhaps in points of singularity of c_1 or c_2 and points in which both these vectors are indeterminate.

The theorem still holds when the vectors c_1, c_2 are proportional. In this case k_1 and k_2 are both solutions of the equation

$$p_i\frac{\partial k}{\partial x} + q_i\frac{\partial k}{\partial y} + r_i\frac{\partial k}{\partial z} = 0, \quad (i = 1, 2),$$

to which both of the equations (3) reduce in form, and γ_1 and γ_2 are both proportional to c_1 and c_2.

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AN ELEMENTARY DERIVATION OF THE PROBABILITY FUNCTION.

BY CAPTAIN ALBERT A. BENNETT, C.A.R.C.

WE shall derive by means of elementary considerations the equation of the probability curve from the sequence of binomial coefficients. If the asymptotic form of $x!$ be obtained, the problem is very simple but none the less merits attention. The asymptotic form of $n!$, viz., $\sqrt{2\pi n}(n/e)^n e^{\theta/(12n)}$, $0 < \theta < 1$, might of course be taken for granted, but so far as is known to the writer, no purely algebraic attempt to prove Stirling's formula has been successful. The proof here given does use ratios of factorial expressions but proves the asymptotic values of these directly. The original derivation by Gauss is elegant and elementary but does not attempt to show the relation of probability in the finite cases to the transcendental probability function $ce^{-h^2x^2}$ nor does that derivation appear to adapt itself to any modification which will render obvious the fact that the infinite case is but a limit obtained from the finite case.

It is true that the method here described makes use of the convergence and the value approached of a single infinite product, viz., Wallis's formula for $\pi/2$,

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots,$$

but this infinite product is a familiar one and is discussed in nearly every textbook treatment of infinite products.

Let a succession of adjacent rectangles R_k with altitudes, $\binom{2m}{m+k}a_m$, and common length of base, b_m, be erected on a straight line. The total area of these rectangles for a given m, with k ranging, is, by the binomial theorem, $2^{2m}a_mb_m$. If a_m and b_m be so varied with m that the middle rectangle R_0 remains of finite height and the area remains finite, then the rectangles form approximations to the probability curve. For the mid-ordinate $\binom{2m}{m}a_m$ to be finite, a_m must be asymptotically proportional to the reciprocal of the middle term $\binom{2m}{m}$: We seek therefore the asymptotic form of the middle term M_m. Let $C_m = mM_m^2/16^m$; then $C_{m+1}/C_m = (2m+1)^2/2m(2m+2)$. Now $C_1 = \frac{1}{4}$, hence, in general, for $m > 1$,

$$C_m=\frac{1}{2}\left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdots\frac{(2m-1)^2}{(2m-2)2m}\right].$$

Hence $M_m \sim 4^m/\sqrt{m\pi}$, and apart from an arbitrary constant factor, $a_m \sim \sqrt{m\pi}/4^m$. Since the area $2^{2m}a_mb_m$ is also to be finite, we have, except for an arbitrary constant factor, $b_m \sim 1/\sqrt{m}$.

If a definite point x be chosen upon the base line, this point will fall within the rectangle numbered k only if kb_m does not differ from x by as much as b_m; the k denoting a fixed x must therefore vary with m, and we have $x \sim kb_m$, or $k \sim x\sqrt{m}$.

The asymptotic ratio of the general term $\binom{2m}{m+k}$ to the middle term $\binom{2m}{m}$ may be written in the form

$$\binom{2m}{m+k}\Big/\binom{2m}{m} = \frac{1}{1+\frac{k}{m}} \cdot \frac{1}{1+\frac{k}{m-1}} \cdots \frac{1}{1+\frac{k}{m-k+1}}$$

$$\therefore \quad \left[\frac{1}{1+\frac{k}{m}}\right]^k > \binom{2m}{m+k}\Big/\binom{2m}{m} > \left[\frac{1}{1+\frac{k}{m-k+1}}\right]^k.$$

Replacing m partially in accordance with $k \sim x\sqrt{m}$, we have

$$\left(\frac{1}{1+x^2/k}\right)^k > \binom{2m}{m+k}\Big/\binom{2m}{m} > \left(\frac{1}{1+x^2/(k-x^2+x^2/k)}\right)^k.$$

Using the fact that the reciprocal of the middle term is asymptotically equal to the altitude factor a_m, we obtain by passing to the limit for m and hence also for k the asymptotic form of the height of the rectangle covering or touching the point x, viz., $y(x) = e^{-x^2}$.

We have thus obtained the equation $y = e^{-x^2}$ as the limit of the altitude of a rectangle contained in a sequence erected proportional to the binomial coefficients, and by different units of proportionality we would have obtained the form $y(x) = ce^{-k^2x^2}$, geometrically no more general. We have incidentally obtained the area $\int_{-\infty}^{+\infty} e^{-x^2}dx = \lim 4^m a_m b_m = \sqrt{\pi}$, directly, without recourse to such devices as polar integration of a double integral. Even the binomial theorem may be interpreted in the limit, giving

$$f(x\sqrt{2}) = \sqrt{\pi/2}\int_{-\infty}^{+\infty} f(x+t)f(x-t)\,dt,$$

where $f(t)$ is e^{-x^2}.

This treatment is believed to be original, but the literature available for examination by the author is that customary to an army post, "somewhere on the Gulf of Mexico,"—nil.

THE INTRINSIC EQUATION FOR EULER'S RESISTANCE INTEGRAL.

BY PROFESSOR G. H. LIGHT.

Euler, in his Scientia Navalis, Proposition 53, gives a problem whose solution leads to the resistance integral

$$I = \int_{t_0}^{t_1} \frac{y'^3}{x'^2 + y'^2}\,dt. \tag{1}$$

This problem is as follows: Among all curves *AM* which with the axis *AP* and perpendicular *PM* comprehend the same area, to find that one which with its symmetric branch on the opposite side of the axis *AP* will form the figure offering the least resistance in water, when it moves in the direction *PA* along the axis.

This is a problem of the isoperimetric type where the integral *I* to be minimized is the integral (1) and the condition imposed is that the area between the axis and the curve is fixed, i. e.,

$$G = \int_{t_0}^{t_1} yx'dt = l.$$

The solution of this problem is known to be

$$x = \frac{1}{\lambda}\,\frac{p^2 - 1}{(p^2 + 1)^2} + \alpha, \quad y = \frac{1}{\lambda}\,\frac{2p^3}{(p^2 + 1)^2} + \beta,$$

where $p = y'/x'$ and α, λ and β are constants. Since this curve can always be obtained by a similarity transformation from the curve

$$x = \frac{p^2 - 1}{(p^2 + 1)^2}, \quad y = \frac{2p^3}{(p^2 + 1)^2}, \tag{2}$$

the general properties of the extremal can be obtained from the latter.

So far as is known, the exact nature of the extremals has never been determined. It is simply stated that they are rational curves of the fourth order with three cusps. The object of this paper is to show that when their intrinsic equation is obtained, it is seen that the curve is a hypocycloid of three cusps. This is deduced in the following way:

The radius of curvature is found from (2) to be

$$R = \frac{2p(3 - p^2)}{(1 + p^2)^{3/2}}. \tag{3}$$

The length of arc is

$$S = \int_0^p \frac{2p(3 - p^2)}{(1 + p^2)^{5/2}}\, dp = -\frac{8}{3}\,\frac{1}{(1 + p^2)^{3/2}} + \frac{2}{(1 + p^2)^{1/2}} + \frac{2}{3},$$

from which

$$\frac{3}{2}\left(S - \frac{2}{3}\right) = \frac{3p^2 - 1}{(1 + p^2)^{3/2}}. \tag{4}$$

In order to eliminate p from equations (3) and (4), form the expressions $R^2/4$ and $\frac{9}{4}(S - \frac{2}{3})^2$. It is seen at once that

$$\frac{R^2}{4} + \frac{9}{4}\left(S - \frac{2}{3}\right)^2 = 1, \quad \text{or} \quad R^2 + 9\left(S - \frac{2}{3}\right)^2 = 4, \tag{5}$$

which is the intrinsic equation of a hypocycloid. The curvature of (2) is

$$k = \frac{(1 + p^2)^{3/2}}{2p(3 - p^2)}, \tag{6}$$

also

$$\frac{dx}{dp} = \frac{2p(3 - p^2)}{(1 + p^2)^3}, \quad \frac{dy}{dp} = \frac{2p^2(3 - p^2)}{(1 + p^2)^3}. \tag{7}$$

From (6) and (7) it is seen that when $p = 0$ or $\pm \sqrt{3}$, dx/dp and dy/dp become zero and $k = \infty$. Hence (5) is the equation of a hypocycloid of three cusps.

UNIVERSITY OF COLORADO.

HERMITE'S WORKS.

Œuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. IV. Paris, Gauthier-Villars, 1917. 8vo. 593 pp.

THE present volume brings to a close the Œuvres of Hermite. It contains about ninety papers arranged chronologically, dating from 1879 and continuing to the year of his death, 1901.

By a strange coincidence, his first and last papers appeared in the initial volumes of new journals. The *Nouvelles Annales de Mathématiques* founded in 1842 contained the first fruits of his budding genius. Nearly sixty years later his last paper was written to lend the support of his great name to a new enterprise, *Le Matematiche pure ed applicate.* It was only a short note, a sort of open-letter to the editor, which terminates with these words: "Prochainement je vous enverrai quelques remarques complémentaires, et une petite Note sur le produit d'une série quelconque par l'exponentielle e^{-x}." He did not live to carry out his benevolent intentions, but it is pleasant to recall that his last efforts were made in behalf of the rising generation and that he wished them to enjoy the encouragement that early publication and recognition alone affords.

The papers of this last volume are for the most part short and treat isolated topics in the elliptic functions, the Γ function, the functions of Legendre, continued fractions, and the theory of numbers. They represent the labors of an aging hero whose exalted official position makes exacting demands on his time and strength, but whose active brain still takes delight in research.

As one looks over the volume, there is much that catches the eye, but the fragmentary nature of most of the papers makes it impossible to give any adequate account of them without undue expenditure of time and space. Perhaps the most noteworthy are two papers on the application of the elliptic functions to the theory of numbers, viz., those beginning page 138, and page 223. They are a continuation of the line of thought which Hermite employed in his celebrated letter to Liouville, "Sur la théorie des fonctions elliptiques et ses applications à l'arithmétique" (1861). Instead of using the factors of the modular equation in the case of complex multiplication to get relations between the class numbers of binary quadratic forms of negative determinant as Kronecker had done, Hermite showed how the comparison of different developments of Θ quotients led to similar results. This method also gave him the number of decompositions of an integer into a sum of three and five squares, a problem rendered celebrated by the researches of Henry Smith and Minkowski.

A paper of general interest is entitled "Sur les racines de la fonction sphérique de seconde espèce," i. e., the roots of

$Q_n(x) = 0$. It was written in 1890, the same year as Klein's paper on the roots of the hypergeometric function $F(\alpha, \beta, \gamma, x)$. Neither author knew of the other's work at the time of publication, moreover the methods employed are totally different. Hermite's work on $Q_n(x) = 0$ led Stieltjes* to attack the same problem from another and more direct point of view. Neither paper seems to have received the attention it deserves.

The many and interesting properties of the Γ function came in for a large share of Hermite's attention. We note as especially important two papers on this subject: the one entitled "Sur une extension de la formule de Stirling," page 378, composed in 1893, the other "Sur la fonction log $\Gamma(a)$," page 412, written in 1895, and dealing with the same problem, viz., the asymptotic expansion of log Γ.

But of all the subjects dear to Hermite in his later years, the elliptic functions without doubt held first place. He knew Jacobi's Fundamenta Nova by heart and he reveled in the endless maze of formulas which to others seem almost an intolerable burden. Here are a few titles dealing with these functions: "Sur la différentiation des fonctions elliptiques par rapport au module," "Sur l'intégration de l'équation différentielle de Lamé," "Sur une proposition de la théorie des fonctions elliptiques," "Sur une formule d'Euler," "Sur une représentation analytique des fonctions au moyen des transcendantes elliptiques." These will suffice to give the reader an idea of the rest.

A feature of this volume will appeal to all mathematicians, as it reveals in no uncertain way Hermite's generous and affectionate nature; we mean the short sketches of the careers of his departed friends and colleagues, Bouquet, Halphen, Kronecker, Kummer, Cayley, Weierstrass and Brioschi. Speaking of Kronecker he says: "La louage se tait devant le deuil de la Science et l'émotion causée dans tout le monde mathématique, par la perte cruelle du grand Géomètre; à ces regrets douloureux, à ces souvenirs d'une vie remplie par tant de travaux et de découvertes, je joins ceux d'une amitié qui a été pendant trente années, l'honneur de ma vie scientifique et que je ne retrouverai plus."

Of Weierstrass he writes: "La vie de notre illustre Confrère a été en entier consacrée à la Science qu'il a servie avec un absolu dévoument. Elle a été longue et comblée d'honneurs;

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* *Annales de Toulouse*, vol. 4 (1890).

mais devant une tombe qui vient de se fermer, nous ne rappelons que son génie et cette universelle sympathie qui s'accorde à la noblesse du caractère. Weierstrass a été droit et bon; qu'il reçoive le suprême hommage plein de regrets et de respect que nous addressons à sa mémoire! Elle vivra aussi longtemps que des esprits avides de vérités consacreront leur efforts aux recherches de l'Analyse, au progrès de la science du Calcul."

Let us quote from one more éloge, that of Brioschi: "J'ai été associé aux travaux de Brioschi; nous avons souvent mis en commun nos efforts; j'ai suivi sa carrière qui a été si belle, remplie par l'étude, et de grands services rendus à son pays. Nul ne ressent plus que moi la perte du grand géomètre, et de l'homme d'honneur, le souvenir de son amitié, d'une étroite liaison remontant à notre jeunesse me restera à jamais l'un des meilleurs et des plus chers de toute ma vie."

Other interesting biographical matter is contained in an address by Hermite at the inauguration of the new Sorbonne in 1900. It gives a brief but illuminating analysis of the work of the chief French mathematicians in the first half of the nineteenth century.

Finally we note the touching discours pronounced by Hermite at the celebration of his seventieth anniversary in 1892. A photograph of the medallion by Chaplain presented to him on this occasion by his friends and admirers from all over the world forms the frontispiece of the volume and a facsimile reproduction of an extract of a letter to Tannery relative to Hermite's classic researches on the elliptic modular function in 1858 brings the volume to a fitting close.

JAMES PIERPONT.

BLICHFELDT'S FINITE COLLINEATION GROUPS.

THE author of "Finite Collineation Groups" is certainly under great obligations to Professor H. H. Mitchell for his very generous and thorough review in this BULLETIN of February, 1918; particularly so since Professor Mitchell had evidently read the book with great care. The author is in full accord with the reviewer on a number of the defects pointed out by the latter; and he offers herewith a few remarks in the hope of clearing up those statements in the book that seemed inexact or obscure to him upon reading Professor Mitchell's review. Unless otherwise stated, page and paragraph references are to the book in question.

1. (BULLETIN, page 247.) Kronecker's theorem was originally stated correctly by the author (*Transactions of the American Mathematical Society*, 1903, pages 390–391; *Mathematische Annalen*, 1907, page 558 (in this place, line 3 from the bottom, the inequality $t < p$ should read $t < p^{n-1}$)). The incorrect form occurs in *Mathematische Annalen*, ibid., page 559; in Finite Groups by Miller-Blichfeldt-Dickson, page 240; and in the book under discussion, page 187. After making the proper corrections in the sentence beginning "Kronecker's theorem implies" (page 187, lines 3–5), namely by substituting for "k roots" the following: "the roots of this equation (either directly or after adding a certain number of pairs of mutually cancelling roots of unity)," the author would leave the next sentence unaltered, and then introduce a new sentence (line 8) to read: "In no case need p be taken greater than k." The remaining portion of 6° will stand as it is. Under 7°, the sentence beginning (line 6 from the bottom) "More definitely" should be replaced by "In particular, if the total number of roots in N equals k, then all these roots are equal, or $N = 0$; if the number of roots is $< k$, then $N = 0$."

2. (BULLETIN, page 249.) In the chapter on group characteristics the meaning of the term "equivalent" is in accordance with the usage in the writings of Burnside, Frobenius, and Schur, though these authors say "equivalent representations of a group" instead of "equivalent groups." Two simply isomorphic groups having their operators so arranged that S_i in the one corresponds to T_i in the other, $i = 1, 2, \cdots, g$, are equivalent if a change of variables (transformation) V exists such that $V^{-1}S_iV = T_i$ for every subscript i. The same two groups may possibly not be equivalent when a new one-to-one correspondence is set up in a different manner. The author has, admittedly, used the word freely in the chapters on the binary, ternary and quaternary groups without calling attention to the fact that in these chapters, if the phrase "non-equivalent" were used in its strict sense, we should have, for example, two non-equivalent primitive binary groups (or two non-equivalent representations of the group) of order 60φ; namely the group (E), page 73, and that group whose trans-

formations are obtained from the corresponding transformations of (E) by interchanging $+\sqrt{5}$ and $-\sqrt{5}$ (this cannot be brought about by a change of variables).

3. (BULLETIN, page 250.) That a primitive group in four variables cannot contain a transformation of order 9 follows from § 117, the reference to this fact being omitted by an oversight. By the reasoning of § 111 it is found that a tentative transformation of order 9 generates a Sylow subgroup of this order, and the presence in a primitive simple group of such a subgroup would exclude all the types given in § 111 except (a) and (g). Since groups of order $2^a \cdot 3^b$ are not simple, the primitive group in question could only be of the kind discussed in § 117.

4. (BULLETIN, page 250.) Concerning the statement "In view of . . . appears superfluous," the author remarks that as yet (§§ 106–108) it had not been shown that T, C, D, V, F generate a *finite* group; i. e., K', containing K as a subgroup, might as yet prove not to be of finite order. Hence the necessity for showing that K' and K are identical, and this is in part the purpose of § 109 (cf. §§ 115, 116, where the same problem is encountered for other groups).

5. The author hereby takes opportunity to emphasize that throughout the chapter on group characteristics, the term "simple group" has its customary meaning; i. e., a group which contains similarity transformations is not "simple" here, though it be given that the corresponding collineation group is simple. (Cf. § 51, 1°, concerning the use of the term in the chapters on groups in 2–4 variables.)

6. In conclusion the author submits the following additional list of errata (the first two were suggested by Professor F. N. Cole):

Page 18, line 1, and page 55, line 16. Read "any set" instead of "any one set."
" 129, " 16. Corollary 2 should read "Let . . . characteristics of two simply isomorphic groups . . ." instead of "Let . . . characteristics of two groups. . . ."
" 70, " 14. The equation $\alpha^{\varphi} = 1$ should be $\alpha^{3\varphi} = 1$.
" 73, " 6 above § 59. After $T'^2 = E_1$, insert: $S'T'$ = a transformation of order 3φ.
" 104, " 5 from bottom. Replace the first x_2' by x_1'.
" 121, last line. Replace $(x_1) = \alpha x_1$ by $(x_1)M = \alpha x_1$.
" 126, line 7 from bottom. Replace f by cf.
" 127, " 8. The letters $\overline{X}_1, \ldots, \overline{X}_n$ should be accented: $\overline{X}_1', \ldots, \overline{X}_n'$.

Page 133, in Exercise 1, after ". . . consists of" insert "the group represented by E repeated 60 times."
" 138, footnote. Replace 6° by 7°.
" 182, line 20. The second of the set of numbers given, namely 6, should be 3.
" 182. At end of footnote marked * write: Jordan's method is to express suitable powers of the solutions of the linear differential equation as rational functions of x and of all the roots of a certain algebraic equation. The degree of this equation will then generally be lower than the corresponding degree found above. See *Crelle*, vol. 84, pp. 93, 112.

H. F. BLICHFELDT.

REMARKS ON ELLIPTIC INTEGRALS.

IT is known that an elliptic integral of the first kind is everywhere one-valued, finite, and continuous on its associated Riemann surface, while the elliptic integral of the second kind is algebraically infinite, and the elliptic integral of the third kind is logarithmically infinite at certain points of the surface. This is a characteristic distinction of these integrals and is essential in their study. It is also true of the hyperelliptic and abelian integrals.

The Legendre form of the integral of the first kind is

$$F(k, \varphi) = \int_0^\phi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.$$

When $k = 1$, this integral becomes

$$F(1, \varphi) = \int_0^\phi \frac{d\varphi}{\cos \varphi} = \log \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

If further $\varphi = \frac{1}{2}\pi$, it is seen that the complete elliptic integral

$$F_1(1) = F\left(1, \frac{\pi}{2}\right)$$

is logarithmically infinite, while $F_1(0) = \pi/2$.

As this is the only possible chance, remote though it be, for an integral of the first kind "to claim kin" with one of the third kind, I don't see why a gentleman from Alabama, where relationships are cherished, the connection often being even more remote, should suffer a "jolt" (see the BULLETIN, February, 1918, page 253) when I write "F_1 increases from $\pi/2$ to logarithmic infinity."

I am glad, however, that the "logarithmic" injection left him so dazed that he did not notice the glaring error found a few pages farther on in the monograph on Elliptic Integrals. I take this opportunity of correcting it. On page 33, line 12 below, is found "in the formulas $\operatorname{sn} iu = i \operatorname{tn}(u, k')$, etc., write $u + iK$ for u." This obviously should be "write $u + K'$ for u" with the resulting fundamental formula

$$\operatorname{cn}(iu + iK', k) = -\frac{1}{k}\frac{\operatorname{dn}(u, k')}{\operatorname{sn}(u, k')}.$$

This is found correctly given in my larger book, page 464, and follows at once from formulas (XIX) and (XVII) of pages 248 and 247 of that work. The formula is also correctly derived in two different ways in my lecture notes, from which I thought the monograph was taken. This leads me to suggest that an author be allowed one bad mistake for every 100 pages, the same to be classified under the heading "inexplicables, lapses, aberrations, etc."

May I also add that in my lectures the pronoun "we" without intentionally implying anything personal is perhaps too frequently used? At any rate one of the editors of the monograph in question thought this to be the case. To oblige him the other editor and I suppressed some of the "we's" and it appears that we did not make other corresponding changes in at least two places. Thus two infelicitous "grammatical connections" remain. I am obliged to Professor Carmichael for not characterizing them more harshly.

HARRIS HANCOCK.

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SHORTER NOTICES.

Differential Calculus. By H. B. PHILLIPS. New York, Wiley, 1916. 162 pp.

Integral Calculus. By H. B. PHILLIPS. New York, Wiley, 1917. 194 pp.

To expound a few central methods and apply them to a large variety of examples to the end that the student may learn principles and gain power is the plan which the writer announces in the preface to this text. A perusal of the book leads to the conclusion that there is justification for this claim. The explanations are brief and simple and the problems offer variety and appear to be well graded. The question of answers is met by furnishing them at the end of each book, for the problems in the chapters, and adding to each a set of over two hundred exercises without answers. Although the differential and integral calculus form two separate books (which are bound separately or together), the arrangement of chapters is such that it would be easy for a teacher who desired to do so to bring in the simpler work in integration before the last chapters on differentiation. The order of subjects in the differential calculus is to be noted. Chapter II is on the derivative and the differential. Some may object that the introduction of the latter before the student has had time to assimilate the idea of the former will cause confusion. There is, however, room for difference of opinion in this matter. After the differentiation of algebraic functions and a brief discussion of rates, there is a chapter on maxima and minima, with simple concrete problems. It is not until after the chapter on differentiation of transcendental functions that there is a consideration of tangents and normals, angles between curves, points of inflection, curvature, etc. The differential calculus is completed by chapters on velocity, series, and partial differentiation. In the book on integral calculus in Chapter II (on formulas and methods of integration) there is a noteworthy omission of those "ingenious devices" that are so dear to some writers of texts and that are so apt to inspire a beginner with the feeling that success in integration is a matter of luck rather than of knowledge of principles. After a short and simple explanation of definite integrals, there are chapters on simple areas and volumes, other geometrical applications, mechanical and physical applications, approximate methods, double and triple integrations, and, finally, a longer chapter on differential equations. The book is concluded by a brief table of integrals and a table of natural logarithms.

E. B. COWLEY.

Modern Business Arithmetic, Complete Course. By H. A. FINNEY and J. C. BROWN. New York, Holt, 1917.

THE book under review is a modern up-to-date text on the arithmetic of modern business, intended primarily for use as a text in commercial schools and business colleges.

The book has a number of features of interest to the mathematician as well as to the teacher of elementary mathematics. On the purely arithmetical side the most important feature is the emphasis placed on drill in correct methods of computation as essential to the acquirement of facility and accuracy, which means efficiency, in practical work. Numerous exercises, both oral and written, to be completed within a certain time limit, are given for the purpose of developing speed. Various checks, such as casting out the nines, and also special short methods are given, and the pupil is taught to check his results at every step. This aspect of the work is especially important in grade work, as most children rely on an answer book or verification by the teacher for the proof of their work, and have no confidence in their own work or means of determining its correctness. This lack of initiative and self-confidence constitutes a serious defect in our present training in arithmetic which the methods adopted in this book would assist materially in correcting.

On its applied side, the work, although comprised in less than 500 pages, is a veritable compendium of modern business practice, and will prove a valuable reference book for teachers of seventh and eighth grade arithmetic. Ruled forms and tabulated business statistics are freely used, in order to put the pupil in touch with the actual demands of the business world.

The first fourteen chapters, covering 152 pages, are devoted to the fundamental processes of arithmetic, including percentage, and the use of graphs for presenting business statistics. Numerous exercises are included, many of which are purely arithmetical, while others are stated in the form of problems met in business practice.

The remainder of the book is devoted to technical commercial applications. Chapters 15–23, covering 79 pages, relate to trading activities, including such topics as profit and loss, recording sales, paying for goods, collecting bills and taking inventory.

Chapters 24–28, comprising 34 pages, are on borrowing and loaning, and include the modern field of contract purchases and installment payments.

Chapters 29–36, covering 54 pages, are devoted to business expenses, such as wages and payrolls, insurance and taxation, and include such modern topics as depreciation, advertising and the income tax.

Chapters 37–60, covering 19 pages, are concerned with business organization, partnership, corporations, insolvency and bankruptcy.

Chapters 41–45, occupying 40 pages, relate to business efficiency, and form the most up-to-date section of the entire book. The nature of this section is apparent from the list of topics covered, which include factory costs, buying and selling expenses, determination of profit and loss by departments and for separate sales, and tabulations for the sales manager such as department and salesmen's records.

Chapters 46–48, occupying 12 pages, conclude the work with miscellaneous topics not previously treated, such as consignments and commissions, life insurance and farm management.

The impression made by this book is that the authors have succeeded in producing not only an unusually complete and well written treatise on business arithmetic, but also a work that is eminently teachable, as it is bright and interesting, and well arranged for classroom purposes.

S. E. SLOCUM.

Plane Trigonometry with Tables. By E. H. BARKER. Philadelphia, P. Blakiston's Son and Co., 1917. 172 pp.

BARKER'S Trigonometry treats, in its few pages, the usual topics of the subject. The functions of an acute angle are defined as ratios in the first chapter, the functions of a general angle being postponed to Chapter IV, while the intervening chapters deal with the relations between the functions of an acute angle. For the graphs the line values of the functions are used. The addition formulas are proved for acute angles, without the use of directed lines. Geometrical proof is given for the laws of sines and cosines, and analytical proof for the law of tangents. Some unusual formulas are given in the work on solutions of triangles. The ninth chapter, on circular measurement of angles and inverse identities, completes the work on trigonometry. The definition, explanation and use of logarithms are given in Chapter X.

Since there is such a variety of trigonometry texts on the market, one looks critically at each new text to see what good points warrant adding one more book to the already large supply. Among the virtues possessed by Barker's Trigonometry we note its small amount of detail in general, reinforced, however, by a wealth of applications in some cases where familiarity with the ideas can be gained only by repeated practice with them. Occasionally the author seems to lose sight of this apparent ideal of brevity, and gives details certainly unnecessary for a college freshman and probably equally unnecessary for any student of trigonometry. Yet, in other places the student is given credit for power which few students of trigonometry would possess.

The author gives some excellent advice to students, especially in urging the fullest use of given material in preference to computed material, in suggesting a reasonably accurate figure as a guide to the solution of a triangle and as a check on results, and in emphasizing checks for results in general. Throughout, the text offers a large number of examples chosen from varied material, and often useful for developing the initiative of the student.

The author has given "special attention to the preparation of the tables and has designed the tabular arrangement to give the computer maximum efficiency with a minimum of labor." In two respects the opposite of this aim seems to be the result of this special attention. First, the entire five digit mantissa is repeated for each number, giving the pages of the logarithms of numbers a most compact, dense appearance which is hard on the eyes of the computer. Second, the natural functions are given in two separate tables, one for sines and cosines, one for tangents and cotangents, thus often requiring double work in securing two functions of the same angle.

The introductory explanation of the theory of logarithms is clearly given with a well chosen minimum of material. However, it is surprising to find advice to the student to approximate with a four-digit number by taking the preceding four-digit number instead of the nearest four-digit number. The use of four-digit approximation is so frequent throughout the book that the student has very little practice in interpolation.

In fact, the book does not at all help the student to realize the ideal expressed by the author, that "the student should at all times strive to get answers as nearly accurate as the tables will permit." Nothing whatever is said about the degree of accuracy warranted by the given statistics of a problem.

Errata are numerous in the text. Some parts of the book are poorly written. There is a dearth of punctuation in general, and of periods in particular. In places the arrangement of the material is very poor, while in other portions of the text the material is well spaced and the page presents its salient points at a glance. The figures and curves are very good.

M. E. WELLS.

Tables numériques usuelles. Par L. ZORETTI. Paris, Gauthier-Villars, 1917. 8vo. 52 pp. 3 fr.

THESE small tables are designed for the use of engineers, students of college and lower grades, and general computers. They consist of multiplication tables for the multipliers 1, 2, $\cdots$, 9 and multiplicands of three figures; together with columns that give for three figure arguments the reciprocal, the square and the common logarithm to four places. A table of trigonometric functions, grades, and radians for every degree and quarter degree follows. Tables of logarithms of certain common constants, and the values of πd and $\frac{1}{4}\pi d^2$ for two figure arguments close the volume. The leaves are thumb-indexed and by proper spacing made otherwise convenient. The type is clear and the paper with no glare. The book is ample for most of the computations involved in the author's general course in mathematics.

JAMES BYRNIE SHAW.

Ueber gewisse Teilbereiche und Erweiterungen von Ringen. By Dr. Phil. ADOLF FRAENKEL. Leipzig und Berlin, Teubner, 1916. 64 pp.

IN the investigations of the arithmetic properties of the algebraic numbers the fundamental notion is that of domain of rationality, in which the four operations of arithmetic, excluding only division by zero, are permissible. Besides these certain other sets, called orders by Dedekind and rings by Hilbert, have been used. In these rings the operations addition, subtraction, and multiplication are permissible, but division is not always possible and, when possible, does not always lead to a unique result.

These rings are special types of groups, where the term group is used in the more general sense of the French edition of the mathematical encyclopedia. The first general investigation into their nature was made by the author of the present work in his dissertation, published in *Crelle's Journal*, volume 145. In this investigation he makes frequent use of the group property and in the proofs of some theorems he shows that the elements under consideration form a group in the ordinary sense and draws his conclusions from known theorems in the theory of groups.

The present work is a further study of certain types of rings, with special reference to certain subsets, and extensions formed by the adjunction of new elements.

Two elements a and b of a ring R are called equivalent if a is divisible by b and b by a. A divisor of zero is an element b such that $a \cdot b = 0$ when $a \neq 0$. An element which is not a divisor of zero is a regular element. A divisor of zero which, aside from regular elements, has no divisor except itself is called a prime divisor. A simple ring is one which contains only one prime divisor.

The work is devoted to the consideration of rings in which multiplication is commutative and in which the following two conditions are satisfied:

I. There are only a finite number of non-equivalent divisors of zero.

II. If $a \cdot b$ is divisible by c, then there exist two elements c_1 and c_2 such that $c = c_1 \cdot c_2$ and a is divisible by c_1 and b by c_2.

Rings which satisfy these conditions are called decomposable, because from the elements of such a ring R, in which there are n non-equivalent prime divisors, we may construct n simple rings $R_1, R_2, \ldots, R_n$, such that no element of R, except zero, is contained in more than one of the rings R_i, the product of two elements from two different rings R_i and R_j is zero, and every element of R has a unique representation as the sum of n elements, one from each of the rings R_i.

This property is of fundamental importance in the theory of the g-adic numbers which constitute a ring such as has been defined above.

A partial ring $\bar{R}$ of a ring R is a set of elements of R, closed with respect to addition, subtraction and multiplication, but in which the equation $ax = b$ has a solution only when it has a solution in R. Since the author has chosen to call a ring a set of elements in which division by regular elements is always possible and a may be a divisor of zero in R but regular in $\bar{R}$, we see that $\bar{R}$ is not necessarily a ring.

A prime ring is a partial ring which does not contain as a subset another partial ring having the same unit element.

The special prime ring is that prime ring whose unit element is the unit element ϵ of R.

A part of the first chapter is devoted to the determination of the structure of the special prime rings. They are found to be of four types according as the sequence ϵ, 2ϵ, 3ϵ, ... contains: (1) neither zero nor divisors of zero; (2) zero but no other divisor of zero; (3) both zero and divisors of zero; (4) divisors of zero but not zero.

The concluding section of the chapter deals with prime rings in general and contains a theorem regarding the number of prime rings in a given ring and their structure.

To aid the reader in understanding the theory the author gives, at various points in the chapter, examples of rings chosen from systems of residues with respect to a certain modulus or from systems of complex numbers.

Chapter II consists of a study of the rational functions of x in a simple ring R. The degree of a polynomial is defined as usual. The order of a polynomial is the exponent of the highest power of x whose coefficient is a regular element of R. The degree of the product of two polynomials may be less than the sum of their degrees, but the order of the product is the sum of the orders.

A polynomial which has at least one regular coefficient is called a primitive polynomial. Every primitive polynomial of order n can be multiplied by a primitive polynomial of order zero so that the product will be of degree n. By applying this theorem the author discusses Euclid's algorithm in a simple ring. This process always leads to the highest common divisor if the successive remainders are all primitive, but fails if the process finally gives a remainder which is not primitive.

Chapter III contains a discussion of the extensions of a simple ring formed by the adjunction of new elements. An extension formed by adjoining an element which satisfies an algebraic equation in R is called an algebraic extension. If the element which is adjoined does not satisfy an algebraic equation in R the extension is called transcendental.

From the simple transcendental extensions formed by the adjunction of the element x to R the author passes to the algebraic extensions by assigning to x a value which is a root of an algebraic equation.

Finally by making use of the theorem regarding the decomposition of rings the results found are extended to rings having more than one prime divisor.

G. E. WAHLIN.

Calcul des Systèmes Elastiques de la Construction. Par ERNEST FLAMARD. Paris, Gauthier-Villars, 1918.

THIS work is a treatise of 200 pages, published under the imprint of the "Encyclopédie Industrielle," on the application of the principle of least work to the calculation of the reactions and deflections of straight and curved beams, the elastic arch, and pin-connected structures having redundant members or supports.

The feature of most interest to the writer of this review is the evidence the work furnishes that there are mathematicians and engineers in France today of such mental poise that they are able to concentrate their attention on a purely theoretical question of method which brings out no new results and has absolutely no relation to the war or the future.

In this work it is first shown that the elastic forces acting on any section of a solid are reducible to three static elements consisting of a bending moment and normal and shearing forces. The expressions for the work of deformation due to these three elements are then derived, as well as Castigliano's well known theorem, giving the linear and angular displacements of the external forces and couples in terms of the partial derivatives of the work of deformation with respect to these elements. This is followed by the derivation of the principle of least work of deformation. These results are then extended to include the effect of change in temperature. In applying the results to beams under vertical loading, however, it is pointed out that the temperature forces are the only external forces acting parallel to the axis of the beam, and may therefore be considered separately and the equation of restraint due to temperature effects written once for all.

The applications begin with the simplest cases of restrained beams carrying a uniform load or a single concentrated load, and by calculation of the work of deformation, the usual results for moments and reactions at the supports, and the linear and angular deflections, are obtained..

Next in order come continuous beams of n spans with fixed ends and carrying either a uniform load in each span or a single concentrated load in one span. The principle of least work is first used to derive the theorem of three moments, which in the most general case takes the form of a quadrature. The ordinary form of this theorem for constant cross section is then deduced from the general expression as a special case, and the expressions for the linear and angular deflections in any segment are obtained in terms of the derivatives of the work of deformation.

The treatment of continuous beams is followed by the application of the same general method to curved beams, that is, beams of given curvature and having a continuous tangent, loaded uniformly over the chord, or horizontal projection, of the curve. The horizontal and vertical reactions and the moments at the supports are first obtained by applying the method of least work. The linear deflection at any point is then determined by applying a given load at this point, calculating the corresponding deflection from the partial derivatives of the work of deformation with respect to this load, and reducing the load to zero in the resulting expression. The angular deflection at any point of the beam is also obtained in the same way by applying a given couple and after using it for evaluating the derivatives of the work of deformation, reducing it to zero in the resulting formula.

Special cases are obtained from the general formulas for a beam bent into a circular arc with constant cross section, and for different conditions of end restraint. The entire process is then repeated for a curved beam carrying a single concentrated load.

The last section is an application of the general methods to statically indeterminate pin-connected systems or jointed linkworks. The treatment is divided into four cases: (1) when the system contains redundant supports; (2) when it contains redundant members; (3) when it includes redundant supports as well as redundant members; (4) mixed systems, such, for example, as a trussed continuous beam.

The method of attack consists in first separating the statically indeterminate system considered into two separate systems such that each is statically determinate. The principle of least work is then applied by equating the derivatives of the total work of deformation with respect to the reactions common to the two systems to zero, and from these conditions determining the common restraints.

To find the deflections, a unit load is placed at any specified point and the deflection is obtained from the partial derivatives of the work of deformation with respect to this unit load. To shorten the process, the work of deformation is expressed in terms of influence numbers which represent the stress in any member due to the various forces acting. These influence numbers are subsequently determined graphically by drawing a separate stress diagram for each of the applied forces or restraints.

The work throughout is mathematically rigorous, and marks a beginning in an important field, as the method of least work, although familiar as a general principle, has not been used to any extent in the theory of elasticity, and promises a complete solution of many problems which have so far received inadequate treatment. S. E. SLOCUM.

Annuaire pour l'An 1918. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1918.

AMONG the "Notices" of the *Annuaire* for 1918 is a timely one by M. J. Renaud, entitled "L'heure en mer." A general revision of our standards of time is one of the by-products of the war. Europe and America have largely adopted the plan of moving the clock forward one hour in summer, thus recognizing the fact that most of our daily acts are much more closely associated with the numerical names of the hours than with the altitude of the sun. Astronomers still begin their day at noon rather than midnight, but here again there is a movement on foot to synchronize the commencements of the civil and astronomical days. The time at sea, where a vessel is continually changing its longitude and therefore its local time, requires different treatment; the older methods also of fixing the "ship's time" require some alteration in view of the advent of wireless telegraphy, which enables the navigator to keep in constant touch with the land. "Local time" requires a knowledge of the ship's position and any record of events at sea is dependent at present for the hour on the accuracy of the local time. Various suggestions are considered by M. Renaud to avoid the difficulty. Perhaps the most practical is that ships' logs be kept in both local and Greenwich time and that all reports be furnished with the latter time only. Zone time is also perfectly practicable. The reader who has travelled much at sea can enjoy a relaxing hour in considering his personal preferences concerning the "how" and "when" of the daily alteration of the clock.

M. Bigourdan gives a full account of the Egyptian calendar; M. Maurice Hamy summarizes our present knowledge of the connection between solar phenomena and terrestrial magnetism and M. Emile Picard has a brief account of the life and work of Darboux. The last of these "Notices" will be read with interest as showing the varied activities of a scholar who is known to most of us solely as a pure mathematician.

Minor changes and improvements in the body of the volume are somewhat more numerous than usual. One useful feature is the addition in the Index of references to articles appearing in the three previous issues. It will be remembered that the growth of the Annuaire has required a division of the subjects, which are now treated in alternate years or at longer intervals. The new Index will save much trouble in searching previous volumes. The principal astronomical events and tidal data for 1919 are to be found in a supplement.

E. W. BROWN.

NOTES.

THE April number (volume 19, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Proof that certain ideals in a cyclotomic realm are principal ideals," by H. H. MITCHELL; "The order of primitive groups (III)," by W. A. MANNING; "On the degree of convergence of Birkhoff's series," by W. E. MILNE; "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points," by C. E. WILDER; "Transformations of applicable conjugate nets of curves on surfaces," by L. P. EISENHART; "A new integral test for the convergence and divergence of infinite series," by R. W. BRINK; "On the reduction of certain differential equations of the second order," by W. D. MACMILLAN; "Invariants of differential configurations in the plane," by E. F. SIMONDS.

THE concluding (June) number of volume 19 of the *Annals of Mathematics* contains: "A class of developments in orthogonal functions," by TOMLINSON FORT; "A formula of polynomial interpolation," by W. G. SIMON; "Plane nets with equal invariants," by G. M. GREEN; "Recent extensions of Descartes' rule of signs," by D. R. CURTISS; "A general form of integral," by P. J. DANIELL; "Elastic stresses in an infinite solid with a spherical cavity," by T. H. GRONWALL.

THE second annual Register of Officers and Members of the Mathematical Association of America has recently appeared as supplement to the May number of the *American Mathematical Monthly*. The Association has now a membership of 1,056, not including institutional members.

AT the meeting of the Edinburgh Mathematical Society held May 10, 1918, the following papers were read: By D. M. Y. SOMMERVILLE: (*a*) "Some applications of spherical geometry to geometrical optics", (*b*) "Note on the minimum duration of twilight"; by T. H. MILLER: "Note on the radical axis of intersecting circles."

IN the announcement of courses to be given at the University of Illinois (June BULLETIN, page 460), Professor G. A. MILLER's course on "Continuous groups" should have been scheduled for the first instead of the second semester. In the second semester Professor Miller will give a course on "Finite groups," which was not included in the announcement.

FEW personal sacrifices of this war can have been greater than that of Professor CAMILLE JORDAN, who has lost three sons and a grandson, killed by the enemy. The sympathies of all American mathematicians will go out to the man to whose teachings they owe so much and whom they all esteem.

THE following university and college teachers of mathematics have recently entered the national military service:

Mr. Ralph Keffer, of Harvard University, has enlisted in the aviation corps. Mr. F. H. Murray, of Harvard University, and Dr. G. A. Pfeiffer, of Princeton University, have joined the national army. Mr. C. H. Parsons, of Columbia University, has joined the ordnance. Assistant Professor Warren Weaver, of Throop College of Technology, is serving in the science and research division of the signal corps at Washington.

Professor H. E. Hawkes, of the department of mathematics of Columbia University, has been made dean of Columbia College, Dean F. P. Keppel having resigned to become third assistant secretary of war. Professor Hawkes has been acting dean during the past year.

Associate professor D. D. Leib, of Connecticut College, has been promoted to a full professorship of mathematics.

Mr. E. S. Lane, of the University of Chicago, has been appointed instructor in mathematics at Rice Institute.

Mr. Alfred Davis, of the Francis W. Parker School at Chicago, has been made head of the department of mathematics at William and Mary College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Beeger (N. G. W. H.). Over eene functie, vooregesteld door eene reeks van Dirichlet. Proefschrift. Groningen, Gebr. Hoitsema, 1916. 8vo. 84 pp..

Bortkiewicz (L. von). Die Iterationen. Ein Beitrag zur Wahrscheinlichkeitstheorie. Berlin, Springer, 1917. 8vo. 12 + 206 pp. Geb. M. 10.00

Huygens (C.). Œuvres complètes de Christian Huygens publiées par la Société Hollandaise des Sciences. 13 tomes in 14. La Haye, Nijhoff, 1888–1916. 4to. Couv. vélin. Fl. 180.00

d'Ocagne (M.). Cours de géométrie pure et appliquée de l'Ecole polytechnique. Tome 2: Cinématique appliquée. Stéréotomie. Statique graphique. Calcul grafo-mécanique. Nomographie. Paris, Gauthier-Villars, 1918. 8vo. 6 + 364 pp. Fr. 18.00

Pringsheim (A.). Vorlesungen über Zahlen- und Funktionenlehre. 1ter Band, 2te Abteilung: Unendliche Reihen mit reellen Gliedern. Leipzig, Teubner, 1916. 8vo. 8 + 222 pp. M. 10.80

Sierpinski (W.). Analiza. Tom 1. Cześć pierwsza: Liczby rzeczywiste i respolone. 14 + 237 pp. Cześć druga: Dzialania nieskonczone. Moskwa, W. Lemana, 1916–1917. 8vo. 4 + 5 rubles

II. ELEMENTARY MATHEMATICS.

Chankin (J.). See Strassburger (A.).

Comberousse (C. de). Trigonométrie rectiligne et sphérique. Tables des valeurs naturelles des fonctions circulaires de minute en minute d'arc avec 6 décimales. 6e édition, entièrement refondue par R. de Montessus de Ballore. Paris, Gauthier-Villars, 1917. 8vo. Fr. 6.00

Fazzari (G.). I numeri reali e l'equazione esponenziale $a^x = b$ per le scuole medie superiori. Palermo, Libreria Scientifica D. Capozzi, 1918. 4 + 75 pp. L. 1.80

Friis-Petersen (F.) og Jessen (J. L. W.). Mellemskolens ny Regnebog. I–IV Hefte. Kϕbenhaven, Gjellerup, 1917.

Helmick (P. S.). Table of $\log_{10} \sec^2 \theta$. (University of Iowa Monograph, No. 4.) Iowa City, University of Iowa, 1917. 16 pp.

Javelot (G.). Comment résoudre les problèmes de géométrie élémentaire. Paris, F. Nathan, 1918. 12mo. Fr. 3.00

Jessen (J. L. W.). See Friis-Petersen (F.).

Montessus de Ballore (R. de). See Comberousse (C. de).

Passano (L. M.). Plane and spherical trigonometry. New York, Macmillan, 1918. 12mo. 16 + 141 pp. Cloth. $1.25

Strassburger (A.) and Chankin (J.). Systematic drill in arithmetic. Books 2 and 3. New York, Longmans, 1918. 8vo. 10 + 180 + 10 + 181 pp. $0.45 + 0.45

III. APPLIED MATHEMATICS.

Becker (K.). See Cranz (C.).

Bigelow (F. H.). A treatise on the sun's radiation and other solar phenomena. New York, Wiley; 1918. 9 + 385 pp. Cloth. $5.00

Bouasse (H.). Optique géométrique élémentaire. Focométrie, optométrie. Paris, Delagrave, 1918. 8vo. 28 + 326 pp. Cartonné. Fr. 20.00

—— Optique géométrique supérieure. Paris, Delagrave, 1918. 8vo. 30 + 316 pp. Cartonné. Fr. 22.00

Boussinesq (J.). Poussée des terres. Recherche des lois générales de l'état ébouleux produit dans un massif de sable par les déformations planes, parallèles à un plan vertical. Paris, Gauthier-Villars, 1917. 4to. 80 pp. Fr. 6.00

Braun (H.). Extra-premiën verblijf van verzekerden buiten Europa. Bekroond antwoord opeen door de Nationale Levensverzekering-Bank uitgeschreven prijsvraag. Rotterdam, Nationale Levensverzekering-Bank, 1915. Gr. 8vo. 130 pp.

COWLEY (W. L.) and LEVY (H.). Aeronautics in theory and experiment. London, Arnold, 1918. 11 + 284 pp. 16s.

CRANZ (C.). Lehrbuch der Ballistik. 1ter Band: Aeussere Ballistik oder Theorie der Bewegung des Geschosses von der Mündung der Waffe ab bis zum Eindringen in das Ziel. Unter Mitwirkung von K. Becker. 2te Auflage. Gleichzeitig 3te und vollständig umgearbeitete Auflage des "Compendiums der theoretischen ausseren Ballistik" von 1896. Leipzig, Teubner, 1917. Gr. 8vo. 16 + 528 pp. M. 19.00

CREHORE (A. C.). The mystery of matter and energy. New York, Van Nostrand, 1917. 11 + 161 pp. $1.00

EXTRAIT de la Connaissance des temps à l'usage des marins. Ephémerides nautiques pour l'an 1919, publiées par le Bureau des Longitudes. Paris, Gauthier-Villars, 1917. 8vo. Fr. 2.40

FÖPPL (A.). Vorlesungen uber technische Mechanik. 1ter Band: Einfuhrung in die Mechanik. 5te Auflage. Leipzig, Teubner, 1917. 8vo. 16 + 431 pp. M. 10.00

GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Tome 1. 2e édition, revue, corrigée et augmentée. Nouveau tirage. Paris, Dunod et Pinat, 1918. 16mo. 8 + 386 pp. Cartonné. Fr. 6.00

GUILLOT (L.). Cours de mécanique. (Programme des écoles nationales d'arts et métiers.) Tome 4: Moteurs à gaz et à pétrole. Paris, Béranger, 1918. Cartonné. Fr. 19.25

HIERNAUX (G.). See GOUARD (E.).

LECORNU (L.). Cours de mécanique professé à l'Ecole polytechnique. Tome 3. Paris, Gauthier-Villars, 1918. 4 + 665 pp. Fr. 25.00

LEVY (H.). See COWLEY (W. L.).

LIVENS (G. H.). The theory of electricity. Cambridge, University Press, 1918. 8vo. 717 pp. 30s.

MALLIK (D. N.). Optical theories: based on lectures delivered before the Calcutta University. Cambridge, University Press, 1917. 6 + 181 pp. 7s. 6d.

MAREC (E.). La technique du croquis et du dessin industriel. Paris, Dunod et Pinat, 1917. 4to. 6 + 125 pp. Fr. 9.00

ROSE (W. N.). Mathematics for engineers. Part 1, including elementary and higher algebra, mensuration and graphs, and plane trigonometry. London, Chapman and Hall, 1918. 14 + 510 pp. 8s. 6d.

SOULEYRE (A.). La structure des planètes. Bône, Thomas, 1917. 8vo. 152 pp.

WARD (L. F.). Glimpses of the cosmos. Volume 6. Period 1897–1912. New York, Putnam, 1918. 34 + 410 pp. $1.75

YOUNG (A. S.). See YOUNG (C. A.).

YOUNG (C. A.). Lessons in astronomy. Revised by A. S. Young. Boston, Ginn, 1917.

TWENTY-SEVENTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

Barrow, D. F. An application of Fourier's series to probability. Read Feb. 24, 1917. *Annals of Mathematics*, ser. 2, vol. 19, No. 2, pp. 96–105; Dec., 1917.

Beatty, S. Derivation of the complementary theorem from the Riemann-Roch theorem. Read April 29, 1916. *American Journal of Mathematics*, vol. 39, No. 3, pp. 257–262; July, 1917.

Bell, E. T. Fourier series for the squares of Hermite's eighteen doubly periodic theta quotients. Read (San Francisco) April 6, 1918. *Messenger of Mathematics*, vol. 47, No. 4, pp. 49–57; Aug., 1917.

—— Numerical functions of [x]. Read (San Francisco) April 6, 1918. *Annals of Mathematics*, ser. 2, vol. 19, No. 3, pp. 210–216; March, 1918.

Bianchi, L. Concerning singular transformations B_k of surfaces applicable to quadrics. Read April 28, 1917. *Transactions of the American Mathematical Society*, vol. 18, No. 3, pp. 379–401; July, 1917.

Birkhoff, G. D. A theorem on series of orthogonal functions with an application to Sturm-Liouville series. Read Oct. 27, 1917. *Proceedings of the National Academy of Sciences*, vol. 3, No. 11, pp. 656–659; Nov., 1917.

Bliss, G. A. Integrals of Lebesgue. Read (Chicago) April 6, 1917. *Bulletin of the American Mathematical Society*, vol. 24, No. 1, pp. 1–47; Oct., 1917.

Blumberg, H. A theorem on semi-continuous functions. Read (Southwestern Section) Dec. 1, 1917. *Bulletin of the American Mathematical Society*, vol. 24, No. 8, pp. 381–383; May, 1918.

Brink, R. W. A new integral test for the convergence and divergence of infinite series. Read April 29, 1916. *Transactions of the American Mathematical Society*, vol. 19, No. 2, pp. 186–204; April, 1918.

Burgess, H. T. Solution of the matrix equation $X^{-1}AX = N$. Read (Chicago) Dec. 22, 1916. *Annals of Mathematics*, ser. 2, vol. 19, No. 1, pp. 30–36; Sept., 1917.

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—— Pierre Laurent Wantzel. Read Sept. 4, 1917. *Bulletin of the American Mathematical Society*, vol. 24, No. 7, pp. 339–347; April, 1918.

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Carmichael, R. D. Note on convergence tests applicable to series converging conditionally. Read (Chicago) April 6, 1917. *Tôhoku Mathematical Journal*, vol. 11, No. 4, pp. 191–199; June, 1917.

—— On the asymptotic character of functions defined by series of the form $\Sigma\, c_n\, g(x+n)$. Read (Chicago) April 21, 1916. *American Journal of Mathematics*, vol. 39, No. 4, pp. 385–403; Oct., 1917.

—— Elementary inequalities for the roots of an algebraic equation. Read Oct. 27, 1917. *Bulletin of the American Mathematical Society*, vol. 24, No. 6, pp. 286–296; March, 1918.

—— Comparison theorems for homogeneous linear differential equations of general order. Read (Chicago) April 6, 1917. *Annals of Mathematics*, ser. 2, vol. 19, No. 3, pp. 159–171; March, 1918.

—— On the representation of functions in series of the form $\Sigma\, c_n\, g(x+n)$. Read (Chicago) April 7, 1917. *American Journal of Mathematics*, vol. 40, No. 2, pp. 113–126; April, 1918.

CARPENTER, A. F. Some fundamental relations in the projective differential geometry of ruled surfaces. Read (Chicago) April 2, 1915, and (San Francisco) Nov. 25, 1916. *Annali di Matematica*, ser. 3, vol. 26, No. 4, pp. 285–300; Oct., 1917.

CHITTENDEN, E. W. On the equivalence of relations $K_{q_1q_2\cdots}$. Read (Chicago) April 3, 1915. *American Journal of Mathematics*, vol. 39, No. 3, pp. 263–271; July, 1917.

—— See PITCHER, A. D.

COBLE, A. B. Point sets and allied Cremona groups (Part III). Read Aug. 4, 1915. *Transactions of the American Mathematical Society*, vol. 18. No, 3, pp. 331–372; July, 1917.

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